Criteria for generalized macroscopic and mesoscopic quantum coherence

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We consider macroscopic, mesoscopic, and "S-scopic" quantum superpositions of eigenstates of an observable and develop some signatures for their existence. We define the extent, or size S of a superposition, with respect to an observable $\hat{x}$, as being the range of outcomes of $\hat{x}$ predicted by that superposition. Such superpositions are referred to as generalized S-scopic superpositions to distinguish them from the extreme superpositions that superpose only the two states that have a difference S in their prediction for the observable. We also consider generalized S-scopic superpositions of coherent states. We explore the constraints that are placed on the statistics if we suppose a system to be described by mixtures of superpositions that are restricted in size. In this way we arrive at experimental criteria that are sufficient to deduce the existence of a generalized S-scopic superposition. The signatures developed are useful where one is able to demonstrate a degree of squeezing. We also discuss how the signatures enable a new type of Einstein-Podolsky-Rosen gedanken experiment.

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I. INTRODUCTION

Since Schrödinger's seminal essay of 1935 [1], in which he introduced his famous cat paradox, there has been a great deal of interest and debate on the subject of the existence of a superposition of two macroscopically distinguishable states. This issue is closely related to the so-called "measurement problem" [2]. Some attempts to solve this problem, such as that of Ghirardi, Rimini, Weber, and Pearle [3], introduce modified dynamics that cause a collapse of the wave function, effectively limiting the size of allowed superpositions.

It thus becomes relevant to determine whether a superposition of states with a certain level of distinguishability can exist experimentally [4]. Evidence [5,6] for quantum superpositions of two distinguishable states has been put forward for a range of different physical systems including superconducting quantum interference devices, trapped ions, optical photons, and photons in microwave high-Q cavities. Signatures for the size of superpositions have been discussed by Leggett [7] and, more recently, by Korsbakken *et al.* [8]. Theoretical work suggests that the generation of a superposition of two truly macroscopically distinct states will be greatly hindered by decoherence [9,10].

Recently [11], we suggested to broaden the concept of detection of macroscopic superpositions, by focusing on signatures that confirm, for some experimental instance, a failure of microscopic or mesoscopic superpositions to predict the measured statistics. This approach is applicable to a broader range of experimental situations based on macroscopic systems, where there would be a macroscopic range of outcomes for some observable, but not necessarily just two that are macroscopically distinct. Recent work by Marquardt *et al.* [12] reports experimental application of this approach.

The paradigmatic example [5,6,13,14] of a macroscopic superposition involves two states ψ_+ and ψ_-, macroscopically distinct in the sense that the respective outcomes of a measurement $\hat{x}$ fall into regions of outcome domain, denoted + and −, that are macroscopically different. We argue in Ref. [11] that a superposition of type

$$\psi_+ + \psi_0 + \psi_-, \tag{1}$$

that involves a range of states but with only some pairs (in this case ψ_+ and ψ_-) macroscopically distinct must also be considered a type of macroscopic superposition (we call these "generalized macroscopic superpositions"), in the sense that it displays a nonzero off-diagonal density matrix element $\langle\psi_+|\rho|\psi_-\rangle$ connecting two macroscopically distinct states, and hence cannot be constructed from microscopic superpositions of the basis states of $\hat{x}$. Such superpositions [15–18] are predicted to be generated in certain key macroscopic experiments, that have confirmed continuous-variable [19–29] squeezing and entanglement, spin squeezing, and entanglement of atomic ensembles [30], and entanglement and violations of Bell inequalities for discrete measurements on multiphoton systems [31–33].

In this paper, we expand on our previous work [11] and derive new criteria for the detection of the generalized macroscopic (or S-scopic) superpositions using continuous variable measurements. These criteria confirm that a macroscopic system cannot be described as any mixture of only microscopic (or s-scopic, where $s<S$) quantum superpositions of eigenstates of $\hat{x}$. We show how to apply the criteria to detect generalized S-scopic superpositions in squeezed and entangled states that are of experimental interest.

The generalized macroscopic superpositions still hold interest from the point of view of Schrödinger's discussion [1] of the apparent incompatibility of quantum mechanics with macroscopic realism. This is so because such superpositions cannot be represented as a mixture of states which give outcomes for $\hat{x}$ that always correspond to one or other (or neither) of the macroscopically distinct regions + and −. The quantum mechanical paradoxes associated with the generalized macroscopic superposition (1) have been discussed in previous papers [11,15,16,34,35].

The criteria derived in this paper take the form of inequalities. Their derivation utilizes the uncertainty principle and the assumption of certain types of mixtures. In this respect they are similar to criteria for inseparability that have been derived by Duan *et al.* [36] and Hofmann and Takeuchi

[37]. Rather than testing for failure of separable states, however, they test for failure of a phase space "macroscopic separability," where it is assumed that a system is always in a mixture (never a superposition) of macroscopically separated states.

We will in this paper note that one can be more general in the derivation of the inequalities, adopting the approach of Leggett and Garg [13] to define a macroscopic reality without reference to any quantum concepts. One may consider a whole class of theories, which we refer to as the "minimum uncertainty theories" (MUTs) and to which quantum mechanics belongs, for which the uncertainty relations hold and the inequalities therefore follow, based on this macroscopic reality. The experimental confirmation of violation of these inequalities will then lead to demonstration of a new type of Einstein-Podolsky-Rosen argument (or "paradox") [38], in which the inconsistency of a type of macroscopic (S-scopic) reality with the completeness of quantum mechanics is revealed [11,34]. A direct analogy exists with the original EPR argument, which is a demonstration of the incompatibility of local realism with the completeness of quantum mechanics [39–41].

II. GENERALIZED *S*-SCOPIC COHERENCE

We introduce in this section the concept of a generalized S-scopic coherence [11], which we define in terms of failure of certain types of mixtures. In the next section, we link this concept to that of the generalized S-scopic superpositions (1).

We consider a system which is in a statistical mixture of two component states. For example, if one attributes probabilities $\wp_1$ and $\wp_2$ to underlying quantum states ρ_1 and ρ_2, respectively (where ρ_i denotes a quantum density operator), then the state of the system will be described as a mixture, which in quantum mechanics is represented as

$$\rho = \wp_1\rho_1 + \wp_2\rho_2. \tag{2}$$

This can be interpreted as "the state is either ρ_1 with probability $\wp_1$, *or* ρ_2 with probability $\wp_2$." The probability for an outcome x of any measurable physical quantity $\hat{x}$ can be written, for a mixture of the type (2), as

$$P(x) = \wp_1 P_1(x) + \wp_2 P_2(x), \tag{3}$$

where $P_i(x)$ $(i=1,2)$ is the probability distribution of x in the state ρ_i.

More generally, in any physical theory, the specification of a state ρ (where here ρ is just a symbol to denote the state, but not necessarily a density matrix) fully specifies the probabilities of outcomes of all experiments that can be performed on the system. If we then have with probability $\wp_1$ a state ρ_1 which predicts for each observable $\hat{x}$ a probability distribution $P_1(x)$ and with probability $\wp_2$ a second state which predicts $P_2(x)$, then the probability distribution for any observable $\hat{x}$ given such mixture is of the form (3). The concept of coherence can now be introduced.

Definition 1. The state of a physical system displays coherence between two outcomes x_1 and x_2 of an observable $\hat{x}$ if and only if the state ρ of the system cannot be considered

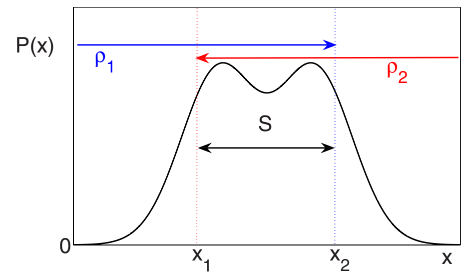

FIG. 1. (Color online) Probability distribution for outcomes x of measurement $\hat{x}$. If x_1 and x_2 are macroscopically separated, then we might expect the system to be described as the mixture (2), where ρ_1 encompasses outcomes $x < x_2$, and ρ_2 encompasses outcomes $x > x_1$. This means an absence of generalized macroscopic coherence, as defined in Sec. II.

a statistical mixture of some underlying states ρ_1 and ρ_2, where ρ_1 assigns probability zero for x_2 and ρ_2 assigns probability zero for x_1.

This definition is independent of quantum mechanics. Within quantum mechanics it implies that the quantum density matrix representing the system cannot be decomposed in the form (2). Thus, for example, $\rho=\wp_+|\psi_+\rangle\langle\psi_+|+\wp_-|\psi_-\rangle\langle\psi_-|$ where $|\psi_\pm\rangle=[|x_1\rangle \pm |x_2\rangle]/\sqrt{2}$ does not display coherence between x_1 and x_2 because it can be rewritten to satisfy Eq. (2). The definition will allow a state to be said to have coherence between x_1 and x_2 if and only if there is no possible ensemble decomposition of that state which allows an interpretation as a mixture (2), so that the system cannot be regarded as being in one or other of the states that can generate at most one of x_1 or x_2. We next define the concept of generalized S-scopic coherence.

Definition 2. We say that the state displays generalized S-scopic coherence if and only if there exist x_1 and x_2 with $x_2 - x_1 \geq S$ (we take $x_2 > x_1$), such that ρ displays coherence between some outcomes $x \leq x_1$ and $x \geq x_2$. This coherence will be said to be macroscopic when S is macroscopic.

If there is no generalized S-scopic coherence, then the system can be described as a mixture (2) where now states ρ_1 and ρ_2 assign nonzero probability only for $x < x_2$ and $x > x_1$, respectively. This situation is depicted in Fig. 1.

An important clarification is needed at this point. It is clearly a vague matter to determine when S is macroscopic. What is important is that we are able to push the boundaries of experimental demonstrations of S-scopic coherence to larger values of S. We will keep the simpler terminology, but the reader might want to understand macroscopic as S-scopic throughout the text.

Generalized macroscopic coherence amounts to a loss of what we will call a generalized macroscopic reality. The simpler form of macroscopic reality that involves only two states macroscopically distinct has been discussed extensively by Leggett [13,14]. This simpler case would be applicable to the situation of Fig. 1 if there were zero probability for result in the intermediate region $x_1 < x < x_2$. Macroscopic reality in this simpler situation means that the system must be in one or other of two macroscopically distinct states ρ_1 and ρ_2 that predict outcomes in regions $x \leq x_1$ and $x \geq x_2$, respectively. The term "macroscopic reality" is used [13] because the definition precludes that the system can be in a superposition of two macroscopically distinct states, prior to measurement. Generalized macroscopic reality applies to the broader situation, where probabilities for outcomes $x_1 < x < x_2$ are not zero, and means that where we have two macroscopically separated outcomes x_1 and x_2, the system can be interpreted as being in one or other of two states ρ_1 and ρ_2, that can predict at most one of x_1 or x_2. Again, the term macroscopic reality is used, because this definition precludes that the system is a superposition of two states that can give macroscopically separated outcomes x_1 and x_2, respectively.

We note that Leggett and Garg [13] define a macroscopic reality in which they do not restrict to quantum states ρ_1 and ρ_2, but allow for a more general class of theories where ρ_1 and ρ_2 can be hidden variable states of the type considered by Bell [42]. Such states are not restricted by the uncertainty relation that would apply to each quantum state, and hence the assumption of macroscopic reality as applied to these theories would not lead to the inequalities we derive in this paper. This point will be discussed in Sec. IV, but the reader should note that the definition of S-scopic coherence within quantum mechanics means that ρ_1 and ρ_2 are quantum states.

III. GENERALIZED MACROSCOPIC AND S-SCOPIC QUANTUM SUPERPOSITIONS

We now link the definition of generalized macroscopic coherence to the definition of generalized macroscopic superposition states [11]. Generally we can express ρ as a mixture of pure states $|\psi_i\rangle$. Thus

$$\rho = \sum_i \wp_i |\psi_i\rangle\langle\psi_i|, \tag{4}$$

where we can expand each $|\psi_i\rangle$ in terms of a basis set such as the eigenstates $|x\rangle$ of $\hat{x}$: $|\psi_i\rangle = \sum_x c_x |x\rangle$.

Theorem A. The existence of coherence between outcomes x_1 and x_2 of an observable $\hat{x}$ is equivalent, within quantum mechanics, to the existence of a nonzero off-diagonal element in the density matrix, i.e., $\langle x_1|\rho|x_2\rangle \neq 0$.

Proof. The proof is given in Appendix A. ■

Theorem B. In quantum mechanics, there exists coherence between outcomes x_1 and x_2 of an observable $\hat{x}$ if and only if in any decomposition (4) of the density matrix, there is a nonzero contribution from a superposition state of the type

$$|\psi_S\rangle = c_{x_1}|x_1\rangle + c_{x_2}|x_2\rangle + \sum_{x \neq x_1, x_2} c_x |x\rangle \tag{5}$$

with $c_{x_1}, c_{x_2} \neq 0$.

Proof. If each $|\psi_i\rangle$ cannot be written in the specific form (5), then each $|\psi_i\rangle\langle\psi_i|$ is either of form ρ_1 or ρ_2, so that we can write ρ as the mixture (2). Hence the existence of coherence, which implies ρ cannot be written as Eq. (2), implies the superposition must always exist in Eq. (4). The converse is also true: if the superposition exists in any decomposition, then there exists an irreducible term in the decomposition that assigns nonzero probabilities to both x_1 and x_2, and therefore the density matrix cannot be written as Eq. (2). ■

We say that a generalized S-scopic superposition of states $|x_1\rangle$ and $|x_2\rangle$ exists when any decomposition (4) must contain a nonzero probability for a superposition (5), where x_1 and x_2 are separated by at least S. Throughout this paper, we define the size of the generalized superposition

$$|\psi\rangle = \sum_k c_k |x_k\rangle \tag{6}$$

(where $|x_k\rangle$ are eigenstates of $\hat{x}$ and each $c_k \neq 0$) to be the range of its prediction for $\hat{x}$, this range being the maximum value of $|x_k - x_j|$ where $|x_k\rangle$ and $|x_j\rangle$ are any two components of the superposition (6) (so $c_k, c_j \neq 0$).

From the above discussions it follows that within quantum mechanics, the existence of generalized S-scopic coherence between x_1 and x_2 (here $|x_2 - x_1| = S$) implies the existence of a generalized S-scopic superposition of type (5), which can be written as

$$|\psi\rangle = c_- \psi_- + c_0 \psi_0 + c_+ \psi_+, \tag{7}$$

where the quantum state ψ_- assigns some nonzero probability only to outcomes smaller than or equal to x_1, the quantum state ψ_+ assigns some nonzero probability only to outcomes larger than or equal to x_2, and the state ψ_0 assigns nonzero probabilities only to intermediate values satisfying $x_1 < x < x_2$. Where S is macroscopic, expression (7) depicts a generalized macroscopic superposition state. In this case then, only the states ψ_- and ψ_+ are necessarily macroscopically distinct. We regain the traditional extreme macroscopic quantum state $c_-\psi_- + c_+\psi_+$ when $c_0 = 0$.

IV. MINIMUM UNCERTAINTY THEORIES

We now follow a procedure similar to that used to derive criteria useful for the confirmation of inseparability [36]. The underlying states ρ_1 and ρ_2 comprising the mixture (2) are themselves quantum states, and so each will satisfy the quantum uncertainty relations with respect to complementary observables. This and the assumption of Eq. (2) will imply a set of constraints, which take the form of inequalities. The violation of any one of these is enough to confirm the observation of a generalized macroscopic coherence—that is, of a generalized macroscopic superposition of type (7).

While our specific aim is to develop criteria for quantum macroscopic superpositions, we present the derivations in as general a form as possible to make the point that experimental violation of the inequalities would imply not only a generalized macroscopic coherence in quantum theory, but a failure of the assumption (3) in all theories which place the system in a probabilistic mixture of two states, which we designate by ρ_1 and ρ_2, and for which the appropriate uncertainty relation holds for each of the states. In this sense, our approach is similar to that of Bell [42], except that the assumption used here of minimum uncertainties for outcomes of measurements would be regarded as more restrictive than the local hidden variable theory assumption on which Bell's theorem is based.

We make this point more specific by defining a whole class of theories, which we refer to as the MUT, that embody the assumption that any state ρ within the theory will predict the same uncertainty relation for the variances of two incom-

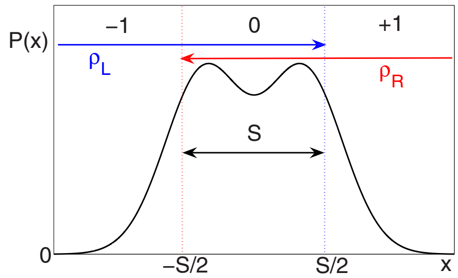

FIG. 2. (Color online) Probability distribution for a measurement $\hat{x}$. We bin results to give three distinct regions of outcome: $0,-1,+1$.

patible observables $\hat{x}$ and $\hat{p}$ as is predicted by quantum mechanics. This is *a priori* not an unreasonable thing to postulate for a theory that may differ from quantum mechanics in the macroscopic regime but agree with all the observations in the well-studied microscopic regime. In this paper we will focus on pairs of observables, such as position and momentum, for which the uncertainty bound is a real number, which with the use of scaling and choice of units will be set to 1, so we can write an uncertainty relation assumed by all MUTs as

$$\Delta^2 x \Delta^2 p \geq 1, \tag{8}$$

where $\Delta^2 x$ and $\Delta^2 p$ are the variances of x and p, respectively. This is Heisenberg's uncertainty relation, and quantum mechanics is clearly a member of MUT. Other quantum uncertainty relations that will be specifically used in this paper include

$$\Delta^2 x + \Delta^2 p \geq 2, \tag{9}$$

which follows for the same choice of units as that of Eq. (8) and has been useful in derivation of inseparability criteria [36].

V. SIGNATURES FOR GENERALIZED *S*-SCOPIC SUPERPOSITIONS: BINNED DOMAIN

In this section we will derive inequalities that follow if there are no s-scopic superpositions (where $s \geqslant S$), so that violation of these inequalities implies existence of an S-scopic superposition (or coherence), as defined in Secs. II and III. The approach is similar to that often used to detect entangled states. Separability implies inequalities such as those derived by Duan *et al.* [36], and their violation thus implies existence of entanglement. This approach has been used to experimentally confirm entanglement, as described in Ref. [22], among others. An experimental description of the approach we use here has been outlined by Marquardt *et al.* [12].

We consider two types of criteria for the detection of a generalized macroscopic superposition (or coherence). The first, of the type considered in Ref. [11], will be considered in this section and uses binned outcomes to demonstrate a generalized S-scopic superposition of states ψ_+ and ψ_- that predict outcomes in specified regions denoted $+1$ and -1 respectively (Fig. 2), where these regions are separated by a minimum distance S. We expand on some earlier results of Ref. [11] for completeness and also introduce more criteria of this type.

A. Single system

Consider a system A and a macroscopic measurement $\hat{x}$ on A, the outcomes of which are spread over a macroscopic range. We partition the domain of outcomes x for this measurement into three regions, labeled $l=-1,0,1$ for the regions $x \leq -S/2$, $-S/2 < x < S/2$, $x \geq S/2$, respectively. The probabilities for outcomes to fall in those regions are denoted $\wp_-$, $\wp_0$, and $\wp_+$, respectively (Fig. 2).

If there is no generalized S-scopic coherence then there is no coherence between outcomes in $l=1$ and $l=-1$, and the state of system A can be written as

$$\rho_{\text{mix}} = \wp_L \rho_L + \wp_R \rho_R, \tag{10}$$

where ρ_L predicts outcomes in the region $x < S/2$, ρ_R predicts outcomes in the region $x > -S/2$, and $\wp_L$ and $\wp_R$ are their respective probabilities. The assumption of this mixture (10) implies

$$P(y) = \wp_L P_L(y) + \wp_R P_R(y). \tag{11}$$

Here y is the outcome of some measurement that can be performed on the system, and $P_{R/L}(y)$ is the probability for a result y when the system is specified as being in state $\rho_{R/L}$. Where the measurement performed is $\hat{x}$, so $y=x$, there is the constraint on Eq. (11) so that $P_R(x)=0$ for $x \leq -S/2$ and $P_L(x)=0$ for $x \geq S/2$.

Now consider an observable $\hat{p}$ (with outcomes p) incompatible with $\hat{x}$, such that the variances are constrained by the uncertainty relation $\Delta^2 x \Delta^2 p \geq 1$. Our goal is to derive inequalities from just two assumptions: first, that $\hat{x}$ and $\hat{p}$ are incompatible observables of quantum mechanics (or of a minimum uncertainty theory), so the uncertainty relation holds for both $\rho_{R/L}$; and, second, that there is no generalized S-scopic coherence.

Violation of these inequalities will imply that one of these assumptions is false. Within quantum mechanics, for which the first assumption is necessarily true, that would imply the existence of a generalized macroscopic superposition of type (7) with outcomes x_1 and x_2 separated by at least S.

If the quantum state is of form (10) or if the theory satisfies Eq. (11), then

$$\Delta^2 p \geq \wp_L \Delta_L^2 p + \wp_R \Delta_R^2 p, \tag{12}$$

where $\Delta^2 p$, $\Delta_L^2 p$, and $\Delta_R^2 p$ are the variances of p in the states ρ_{mix}, ρ_L, and ρ_R, respectively. This follows simply from the fact the variance of a mixture cannot be less than the average variance of its component states. Specifically, if a probability distribution for a variable z is of the form $P(z)=\Sigma_{i=1}^N \wp_i P_i(z)$, then $\Delta^2 z = \Sigma_{i=1}^N \wp_i \Delta_i^2 z + \frac{1}{2}\Sigma_{i \neq i'} \wp_i \wp_{i'} (\langle z \rangle_i - \langle z \rangle_{i'})^2$.

We can now, using Eq. (12) and the Cauchy-Schwarz inequality, derive a bound for a particular function of variances that will apply if the system is describable as the mixture Eq. (10)

$$(\wp_L\Delta_L^2x+\wp_R\Delta_R^2x)\Delta^2p \geq \left[\sum_{i=L,R}\wp_i\Delta_i^2x\right]\left[\sum_{i=L,R}\wp_i\Delta_i^2p\right] \geq \left[\sum_{i=L,R}\wp_i\Delta_ix\Delta_ip\right]^2 \geq 1. \tag{13}$$

The left-hand side is not directly measurable, since it involves variances of $\hat{x}$ in two states which have overlapping ranges of outcomes. We must derive an upper bound for $\Delta_{L/R}^2x$ in terms of measurable quantities. For this we partition the probability distribution $P_R(x)$ according to the outcome domains $l=0,1$, into normalized probability distributions $P_{R0}(x)\equiv P_R(x|x<S/2)$ and $P_+(x)\equiv P_R(x|x\geq S/2)$:

$$P_R(x)=\wp_{R0}P_{R0}(x)+\wp_{R+}P_+(x). \tag{14}$$

Here $\wp_{R+}=\int_{S/2}^{\infty}P_R(x)dx=\wp_+$ and $\wp_{R0}=\int_0^{S/2}P_R(x)dx$. It follows that $\Delta_R^2x=\wp_{R0}\Delta_{R0}^2x+\wp_{R+}\Delta_+^2x+\wp_{R0}\wp_{R+}(\mu_+-\mu_{R0})^2$, where $\mu_+(\Delta_+^2x)$ and μ_{R0} (Δ_{R0}^2x) are the averages (variances) of $P_+(x)$ and $P_{R0}(x)$, respectively. Using the bounds $\wp_{R0}\leq\wp_0/(\wp_0+\wp_+)$, $\Delta_{R0}^2x\leq S^2/4$, $\wp_{R+}\leq 1$, and $0\leq\mu_+-\mu_{R0}\leq\mu_++S/2$, we derive

$$\Delta_R^2x \leq \Delta_+^2x+\frac{\wp_0}{\wp_0+\wp_+}[(S/2)^2+(\mu_++S/2)^2] \tag{15}$$

and, by similar reasoning,

$$\Delta_L^2x \leq \Delta_-^2x+\frac{\wp_0}{\wp_0+\wp_-}[(S/2)^2+(\mu_--S/2)^2]. \tag{16}$$

Here $\mu_\pm$ and $\Delta_\pm^2x$ are the mean and variance of the measurable $P_\pm(x)$, which, since the only contributions to the regions $+$ and $-$ are from $P_R(x)$ and $P_L(x)$ respectively, are equal to the normalized $+$ and $-$ parts of $P(x)$, so that $P_+(x)=P(x|x\geq S/2)$ and $P_-(x)=P(x|x\leq -S/2)$. We substitute Eq. (15) in Eq. (13), and use $\wp_0+\wp_+\geq\wp_R$ and $\wp_0+\wp_-\geq\wp_L$ to derive the final result which is expressed in the following theorem.

Theorem 1. The assumption of no generalized S-scopic coherence between outcomes in regions $+1$ and -1 of Fig. 2 (or, equivalently, of no generalized S-scopic superpositions involving two states ψ_- and ψ_+ predicting outcomes for $\hat{x}$ in the respective regions $+1$ and -1) will imply the uncertainty relations

$$(\Delta_{\text{ave}}^2x+\wp_0\delta)\Delta^2p \geq 1 \tag{17}$$

and

$$\Delta_{\text{ave}}^2x+\Delta^2p \geq 2-\wp_0\delta, \tag{18}$$

where we define $\Delta_{\text{ave}}^2x=\wp_+\Delta_+^2x+\wp_-\Delta_-^2x$ and $\delta\equiv\{(\mu_++S/2)^2+(\mu_--S/2)^2+S^2/2\}+\Delta_+^2x+\Delta_-^2x$. Thus, the violation of either one of these inequalities implies the existence of a generalized S-scopic quantum superposition, and in this case the superposition involves states ψ_+ and ψ_- predicting outcomes for $\hat{x}$ in regions $+1$ and -1, of Fig. 2, respectively.

As illustrated in Fig. 2, the $\Delta_\pm^2x$ and $\mu_\pm$ are the variance and mean of $P_\pm(x)$, the normalized distribution over the domain $l=\pm 1$. $\wp_\pm$ is the total probability for a result x in the domain $l=\pm 1$, while $\wp_0=1-(\wp_++\wp_-)$. The measurement of the probability distributions for $\hat{x}$ and $\hat{p}$ are all that is required to determine whether violation of the inequality (17) or (18) occurs. Where $\hat{x}$ and $\hat{p}$ correspond to optical field quadratures, such distributions have been measured, for example, by Smithey *et al.* [43].

Proof. The assumption of no such generalized S-scopic superposition implies Eq. (10). We have proved that Eq. (17) follows. To prove Eq. (18), we start from Eq. (10) and the uncertainty relation (9), and derive a bound that will apply if the system is describable as Eq. (10): $(\wp_L\Delta_L^2x+\wp_R\Delta_R^2x)+\Delta^2p\geq[\sum_{i=L,R}\wp_i\Delta_i^2x]+[\sum_{i=L,R}\wp_i\Delta_i^2p]\geq[\sum_{i=L,R}\wp_i[\Delta_i^2x+\Delta_i^2p]\geq 2$. Using Eqs. (15) and (16) and $\wp_0+\wp_+\geq\wp_R$ and $\wp_0+\wp_-\geq\wp_L$ we get the final result. ■

B. Bipartite systems

One can derive similar criteria where we have a system comprised of two subsystems A and B. In this case, a reduced variance may be found in a combination of observables from both subsystems. A common example is where there is a correlation between the two positions X^A and X^B of subsystems A and B, respectively, and also between the two momenta P^A and P^B. Such correlation was discussed by Einstein, Podolsky, and Rosen [38] and is called EPR correlation. If a sufficiently strong correlation exists, it is possible that both the position difference X^A-X^B and the momenta sum P^A+P^B will have zero variance.

Where we have two subsystems that may demonstrate EPR correlation, we may construct a number of useful complementary measurements that may reveal generalized macroscopic superpositions. The simplest situation is where we again consider superpositions with respect to the observable X^A of system A. Complementary observables include observables of the type

$$\tilde{P}=P^A-gP^B, \tag{19}$$

where g is an arbitrary constant and P^B is an observable of system B. We denote the outcomes of measurements X^A, P^A, P^B, $\tilde{P}$ by the lower case symbols x^A, p^A, p^B, $\tilde{p}$, respectively. The Heisenberg uncertainty relation is

$$\Delta^2x^A\Delta_{\text{inf},L}^2p^A=\Delta^2x^A\Delta^2\tilde{p} \geq 1. \tag{20}$$

We have introduced $\Delta_{\text{inf},L}^2p^A=\Delta^2\tilde{p}$ so that a connection is made with notation used previously in the context of demonstration of the EPR paradox [44,41]. More generally [39,41], we define an inference variance

$$\Delta_{\text{inf}}^2p^A=\sum_{p^B}P(p^B)\Delta^2(p^A|p^B), \tag{21}$$

which is the average conditional variance for P^A at A given a measurement of P^B at B. The $\Delta^2(p^A|p^B)$ are the variances of the conditional probability distributions $P(p^A|p^B)$. We note that $\Delta_{\text{inf},L}^2p^A$ is the linear regression estimate of $\Delta_{\text{inf}}^2p^A$, but that we have $\Delta_{\text{inf}}^2p^A=\Delta_{\text{inf},L}^2p^A$ for the case of Gaussian states [41]. The uncertainty relation

$$\Delta^2x^A\Delta_{\text{inf}}^2p^A \geq 1 \tag{22}$$

and also $\Delta^2p^A\Delta_{\text{inf}}^2x^A\geq 1$, holds true for all quantum states [35], so that we can interchange $\Delta_{\text{inf}}^2p^A$ with $\Delta_{\text{inf},L}^2p^A$ in the proofs and theorems below.

Theorem 2. Where we have a system comprised of subsystems A and B, the absence of generalized S-scopic superpositions with respect to the measurement X^A implies

$$(\Delta^2_{\mathrm{ave}}x^A + \wp_0\delta)\Delta^2_{\mathrm{inf}}p^A \geq 1. \tag{23}$$

$\Delta^2_{\mathrm{ave}}x^A$, $\wp_0$, and δ are defined as for Theorem 1 for the distribution $P(x^A)$. $\Delta^2_{\mathrm{inf}}p^A$ is defined by Eq. (21) and involves measurements performed on both systems A and B. The inequality (23) also holds replacing $\Delta^2_{\mathrm{inf}}p^A$ with $\Delta^2_{\mathrm{inf},L}p^A$ which is defined by Eq. (20). Thus violation of Eq. (23) implies the existence of the generalized S-scopic superposition, involving states predicting outcomes for X^A in regions $+1$ and -1.

Proof. The proof follows in identical fashion to that of Theorem 1, except in this case the ρ_L and ρ_R of Eq. (10) are states of the composite system, and there is no constraint on these except that the domain for outcomes of X^A is restricted as specified in the definition of $\rho_{R/L}$. The expansion (4) for the density matrix as a mixture is $\rho=\Sigma_r\wp_r|\psi_r\rangle\langle\psi_r|$ where now $\psi_r=\Sigma_{i,j}c_{i,j}|x_i\rangle_A|x_j\rangle_B$, $|x_j\rangle_B$ being eigenstates of an observable of system B that form a basis set for states of B. The generalized superposition (5) thus becomes in this bipartite case

$$|\psi_r\rangle = c_1|x_1\rangle_A|u_1\rangle_B + c_2|x_2\rangle_A|u_2\rangle_B + \sum_{i\neq 1,2} c_{ij}|x_i\rangle_A|x_j\rangle_B, \tag{24}$$

where $|u_1\rangle$ and $|u_2\rangle$ are pure states for system B. If we assume no generalized S-scopic superposition, then ρ can be written without contribution from a state of form (24) and we can write ρ as Eq. (10). The constraint (10) implies $P(\tilde{p})=\Sigma_{I=R,L}\wp_I P_I(\tilde{p})$ where $P_{R|L}(\tilde{p})$ is the probability distribution of $\tilde{p}$ for state $\rho_{R/L}$. Thus Eq. (12) also holds for $\tilde{p}$ replacing p, as do all the results (14)–(16) involving the variances of x^A. Also, Eq. (12) holds for $\Delta^2_{\mathrm{inf}}p^A$ (see Appendix B). Thus we prove Theorem 2 by following Eqs. (12)–(17). ■

In order to violate the inequality (23), we would look to minimize $\Delta^2_{\mathrm{inf}}p^A$, or $\Delta^2_{\mathrm{inf},L}p^A=\Delta^2\tilde{p}$. For the optimal EPR states, P^A+P^B has zero variance, and one would choose for $\tilde{P}$ the case of $g=-1$, so that $\tilde{p}=p^A+p^B$, where p^B is the result of measurement of P^B at B. This case gives $\Delta^2_{\mathrm{inf}}p^A=0$. More generally for quantum states that are not the ideal case of EPR, our choice of $\tilde{p}$ becomes so as to optimize the violation of Eq. (23) and will depend on the quantum state considered. This will be explained further in Sec. VIII.

A second approach is to use as the macroscopic measurement a linear combination of observables from both systems A and B, so, for example, we might have $\hat{x}=(X^A+X^B)/\sqrt{2}$ and $\hat{p}=(P^A+P^B)/\sqrt{2}$. Relevant uncertainty relations include (based on $|[X^A,P^A]|=2$ which gives $\Delta x^A\Delta p^A=1$)

$$\Delta(x^A+x^B)\Delta(p^A+p^B) \geq 2 \tag{25}$$

and

$$\Delta^2(x^A+x^B)+\Delta^2(p^A+p^B) \geq 4, \tag{26}$$

and from these we can derive criteria for generalized S-scopic coherence and superpositions.

Theorem 3. The following inequalities if violated will imply existence of generalized S-scopic superpositions

$$\left[\Delta^2_{\mathrm{ave}}\left(\frac{x^A+x^B}{\sqrt{2}}\right)+\wp_0\delta\right]\Delta^2\left(\frac{p^A+p^B}{\sqrt{2}}\right) \geq 1 \tag{27}$$

and

$$\Delta^2_{\mathrm{ave}}\left(\frac{x^A+x^B}{\sqrt{2}}\right)+\Delta^2\left(\frac{p^A+p^B}{\sqrt{2}}\right) \geq 2-\wp_0\delta. \tag{28}$$

We write in terms of the normalized quadratures so that, following Eq. (25), $\Delta^2(\frac{x^A+x^B}{\sqrt{2}})<1$ would imply squeezing of the variance below the quantum noise level. The quantities $\Delta^2_{\mathrm{ave}}x$, $\wp_0$, and δ are defined as for Theorem 1, but we note that $P(x)$ in this case is the distribution for $\hat{x}=(X^A+X^B)/\sqrt{2}$. S now refers to the size of the superposition of $(X^A+X^B)/\sqrt{2}$.

Proof. In this case the $\rho_{R/L}$ of Eq. (10) are defined as specified originally in Eq. (10) but where x is now defined as the outcome of the measurement $\hat{x}=(X^A+X^B)/\sqrt{2}$. The failure of the form (10) for ρ is equivalent to the existence of a generalized superposition of type (24) where now $|x_i\rangle$ refers to eigenstates of X^A+X^B. Thus the eigenstates $|x_i\rangle$ are of the general form $|x_i\rangle=\Sigma_{x_j}c_j|x_j\rangle_A|x_i-x_j\rangle_B$. The mixture (10) implies Eq. (12) where now p refers to the outcome of $\hat{p}=(P^A+P^B)\sqrt{2}$, and will imply a similar inequality for $\hat{x}$. Application of uncertainty relation (25) for the products can be used in Eq. (13), and the proof of theorem follows as in Eqs. (12)–(17) of theorem 1. The second result follows by applying the procedure for proof of Eq. (18) but using the sum uncertainty relation (26). ■

VI. SIGNATURES OF NONLOCATABLE GENERALIZED S-SCOPIC SUPERPOSITIONS

A second set of criteria will be developed, to demonstrate that a generalized S-scopic superposition exists, so that two states comprising the superposition predict respective outcomes separated by at least size S, but in this case there is the disadvantage that no information is obtained regarding the regions in which these outcomes lie.

This lack of information is compensated by a far simpler form of the inequalities and increased sensitivity of the criteria. For pure states, a measurement of squeezing Δp implies a state that when written in terms of the eigenstates of x is a superposition such that $\Delta x\geq 1/\Delta p$. With increasing squeezing, the extent S of the superposition increases. To develop a simple relationship between S and Δp for mixtures, we assume that there is no such generalized coherence between any outcomes of $\hat{x}$ separated by a distance larger than S. This approach gives a simple connection between the minimum size of a superposition describing the system and the degree of squeezing that is measured for this system. The drawback is the loss of direct information about the location (in phase space for example) of the superposition. We thus refer to these superpositions as "nonlocatable."

A. Single systems

We consider the outcome domain of a macroscopic observable $\hat{x}$ as illustrated in Fig. 3, and address the question of whether this distribution could be predicted from micro-

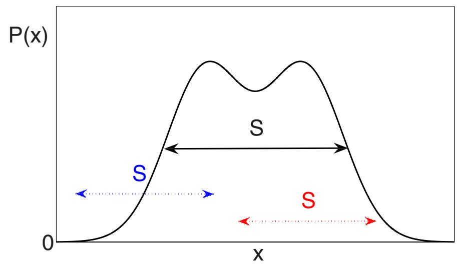

FIG. 3. (Color online) We consider an arbitrary probability distribution for a measurement $\hat{x}$ that gives a macroscopic range of outcomes.

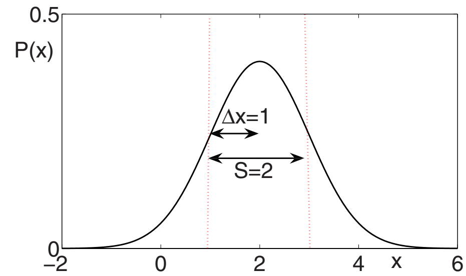

FIG. 4. (Color online) $P(x)$ for a coherent state $|\alpha\rangle$: $\Delta x = \Delta p = 1$.

scopic, or s-scopic ($s < S$), superpositions of eigenstates of $\hat{x}$ alone. The assumption of no generalized S-scopic coherence (between any two outcomes of the domain for $\hat{x}$) or, equivalently, the assumption of no generalized S-scopic superpositions, with respect to eigenstates of $\hat{x}$, means that the state can be written in the form

$$\rho_S = \sum_i \wp_i \rho_{Si}. \tag{29}$$

Here each ρ_{Si} is the density operator for a pure quantum state that is not such a generalized S-scopic superposition, so that ρ_{Si} has a range of possible outcomes for $\hat{x}$ separated by less than S. Hence $\rho_{Si} = |\psi_{Si}\rangle\langle\psi_{Si}|$ where

$$|\psi_{Si}\rangle = \sum_k c_k |x_k\rangle \tag{30}$$

but the maximum separation of any two states $|x_k\rangle, |x_{k'}\rangle$, involved in the superposition (that is with $c_k, c_{k'} \neq 0$) is less than S, so $|x_k - x_{k'}| < S$.

Assumption (29) will imply a constraint on the measurable statistics, namely, that there is a minimum level of uncertainty in the prediction for the complementary observable $\hat{p}$. The variances of each ρ_{Si} must be bounded by

$$\Delta^2_{Si} x < \frac{S^2}{4}. \tag{31}$$

It is also true that

$$\Delta^2 p \geq \sum_i \wp_i \Delta^2_{Si} p. \tag{32}$$

Now the Heisenberg uncertainty relation applies to each ρ_{Si} (the inequality also applies to the MUT's discussed in Sec. IV) so for the incompatible observables $\hat{x}$ and $\hat{p}$

$$\Delta^2_{Si} x \Delta^2_{Si} p \geq 1. \tag{33}$$

Thus a lower bound on the variance of p follows:

$$\Delta^2 p \geq \sum_i \wp_i \Delta^2_{Si} p, \tag{34}$$

$$\geq \sum_i \wp_i \frac{1}{\Delta^2_{Si} x} > \frac{4}{S^2}.$$

We thus arrive at the following theorem.

Theorem 4. The assumption of no generalized S-scopic coherence in $\hat{x}$ will imply the following inequality for the variance of outcomes of the complementary observable $\hat{p}$

$$\Delta p > \frac{2}{S}. \tag{35}$$

The main result of this section follows from Theorem 4 and is that the observation of a squeezing Δp in $\hat{p}$ such that

$$\Delta p \leq 2/S \tag{36}$$

will imply the existence of an S-scopic superposition

$$c_x |x\rangle + c_{x+S} |x + S\rangle + \cdots , \tag{37}$$

namely, of a superposition of eigenstates $|x\rangle$ of $\hat{x}$, that give predictions for $\hat{x}$ with a range of at least S. The parameter S gives a minimum extent of quantum indeterminacy with respect to the observable $\hat{x}$. Here c_x and c_{x+S} represent nonzero probability amplitudes.

In fact, using our criterion (36) squeezing in p ($\Delta p < 1$) will rule out any expansion of the system density operator in terms of superpositions of $|x\rangle$ with $S \leq 2$ (Fig. 4). Thus onset of squeezing is evidence of the onset of quantum superpositions of size $S > 2$, the size $S = 2$ corresponding to the vacuum noise level. This noise level may be taken as a level of reference in determining the relative size of the superposition. The experimental observation [29] of squeezing levels of $\Delta p \approx 0.4$ confirms superpositions of size at least $S = 5$.

B. Bipartite systems

For composite systems comprised of two subsystems A and B upon which measurements X^A, P^A, X^B, P^B can be performed, the approach of the previous section leads to the following theorems.

Theorem 5a. The assumption of no generalized S-scopic coherence with respect to X^A implies

$$\Delta_{\inf} p^A > \frac{2}{S}. \tag{38}$$

$\Delta^2_{\inf} p^A$ is defined as in Eq. (21). The result also holds on replacing $\Delta^2_{\inf} p$ with $\Delta^2_{\inf,L} p$ as defined in Eq. (20).

Theorem 5b. The assumption of no generalized S-scopic coherence with respect to $\hat{x} = (X^A + X^B)/\sqrt{2}$ implies

$$\Delta\left(\frac{p^A + p^B}{\sqrt{2}}\right) > \frac{2}{S}. \tag{39}$$

Proof. The proofs follow as for Theorem 4, but using the uncertainty relations (20) and (25) in Eq. (34) instead of Eq. (33). ∎

The observation of squeezing such that Eq. (38) is violated will imply the existence of an S-scopic superposition

$$c_x|x\rangle_A|u_1\rangle_B + c_{x+S}|x+S\rangle_A|u_2\rangle_B + \cdots \ , \tag{40}$$

namely, of a superposition of eigenstates $|x\rangle_A$ that give predictions for X^A separated by at least S. Similarly, the observation of two-mode squeezing such that Eq. (39) is violated will imply existence of an S-scopic superposition of eigenstates of the normalized position sum $(X^A+X^B)/\sqrt{2}$.

VII. CRITERIA FOR GENERALIZED S-SCOPIC COHERENT STATE SUPERPOSITIONS

The criteria developed in the previous section may be used to rule out that a system is describable as a mixture of coherent states, or certain superpositions of them. If a system can be represented as a mixture of coherent states $|\alpha\rangle$ the density operator for the quantum state will be expressible as

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{41}$$

which is, since $P(\alpha)$ is positive for a mixture, the Glauber-Sudarshan P representation [45]. The quadratures $\hat{x}$ and $\hat{p}$ are defined as $x=a+a^\dagger$ and $p=(a-a^\dagger)/i$, so that $\Delta x=\Delta p=1$ for this minimum uncertainty state, where here a, $a^\dagger$ are the standard boson creation and annihilation operators, so that $a|\alpha\rangle=\alpha|\alpha\rangle$. Proving failure of mixtures of these coherent states would be a first requirement in a search for macroscopic superpositions, since such mixtures expand the system density operator in terms of states with equal yet minimum uncertainty in each of x and p, that therefore do not allow significant macroscopic superpositions in either.

The coherent states form a basis for the Hilbert space of such bosonic fields, and any quantum density operator can thus be expanded as a mixture of coherent states or their superpositions. It is known [46] that systems exhibiting squeezing ($\Delta p<1$) cannot be represented by the Glauber-Sudarshan representation, and hence onset of squeezing implies the existence of some superposition of coherent states. A next step is to rule out mixtures of s_α -scopic superpositions of coherent states. To define what we mean by this, we consider superpositions

$$|\psi_{s_\alpha}\rangle = \sum_i c_i|\alpha_i\rangle, \tag{42}$$

where for any $|\alpha_i\rangle$, $|\alpha_j\rangle$ such that $c_i,c_j\neq 0$, we have $|\alpha_i-\alpha_j|\leq s_\alpha$ for all i, j (s_α is a positive number). We note that for a coherent state $|\alpha\rangle$, $\langle x\rangle=2\alpha$. Thus the separation of the states with respect to $\hat{x}$ is defined as $S_\alpha=2s_\alpha$. The "separation" of the two coherent states $|-\alpha\rangle$ and $|\alpha\rangle$ (where α is real) in terms of x corresponds to $S_\alpha=4\alpha=2s_\alpha$, as illustrated in Fig. 5.

We next ask whether the density operator for the system can be described in terms of the s_α-scopic coherent superpositions, so that

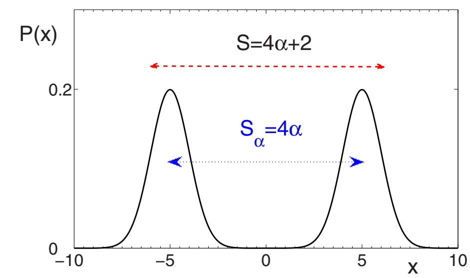

FIG. 5. (Color online) (a) $P(x)$ for a superposition of coherent states $(1/\sqrt{2})\{e^{i\pi/4}|-\alpha\rangle+e^{-i\pi/4}|\alpha\rangle\}$ (here the scale is such that $\Delta x=1$ for the coherent state $|\alpha\rangle$).

$$\rho = \sum_r \wp_r|\psi^r_{s_\alpha}\rangle\langle\psi^r_{s_\alpha}|, \tag{43}$$

where each $|\psi^r_{s_\alpha}\rangle$ is of the form (42). Each $|\psi^r_{s_\alpha}\rangle$ predicts a variance in x which has an upper limit given by that of the superposition $(1/\sqrt{2})\{e^{i\pi/4}|-s_\alpha/2\rangle+e^{-i\pi/4}|s_\alpha/2\rangle\}$. This state predicts a probability distribution $P(x)=\frac{1}{2}\Sigma_\pm P_{G\pm}(x)$, where

$$P_{G\pm}(x) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{-(x\mp s_\alpha)^2}{2}\right] \tag{44}$$

(Fig. 5), which corresponds to a variance $\Delta^2x=\langle x^2\rangle=1+s_\alpha^2=1+S_\alpha^2/4$. This means each $|\psi^r_{s_\alpha}\rangle$ is constrained to allow only $\Delta^2x\leq 1+s_\alpha^2$, which implies for each $|\psi^r_{s_\alpha}\rangle$ a lower bound on the variance Δ^2p so that $\Delta^2p\geq 1/\Delta^2x\geq 1/(1+s_\alpha^2)$. Thus using the result for a mixture (43), we get that if indeed Eq. (43) can describe the system, the variance in p is constrained to satisfy $\Delta^2p\geq 1/(1+s_\alpha^2)$.

Thus observation of squeezing $\Delta^2p<1$, so that the inequality

$$\Delta^2p < 1/(1+s_\alpha^2) \tag{45}$$

is violated, will allow deduction of superpositions of coherent states with separation at least s_α. This separation corresponds to a separation of $S_\alpha=2s_\alpha$ in x between the two corresponding Gaussian distributions (Fig. 5), on the scale where $\Delta^2x=1$ is the variance predicted by each coherent state.

We note that measured values of squeezing $\Delta p\approx 0.4$ [29] would imply $s_\alpha\gtrsim 2.2$. This confirms the existence of a superposition of type

$$|\psi_S\rangle = \sum_i c_i|\alpha_i\rangle = c_-|-\alpha_0\rangle + \cdots + c_+|\alpha_0\rangle, \tag{46}$$

where a separation of at least $s_\alpha=|\alpha_i-\alpha_j|=2.2$ occurs between two coherent states comprising the superposition, so that we may write $\alpha_0=1.1$. Note we have defined reference axes in phase space selected so that the x axis is the line connecting the two most separated states $|\alpha_i\rangle$ and $|\alpha_j\rangle$ so that $|\alpha_i-\alpha_j|=2\alpha_0$ and the p axis cuts bisects this line. Equation (46) can be compared with experimental reports [6] of generation of extreme coherent superpositions of type $(1/\sqrt{2})\{e^{i\pi/4}|-\alpha_0\rangle+e^{-i\pi/4}|\alpha_0\rangle\}$, where $|\alpha_0|^2=0.79$, implying $\alpha_0=0.89$. The corresponding generalized s_α-scopic superposition (46) as confirmed by the squeezing measurement in-

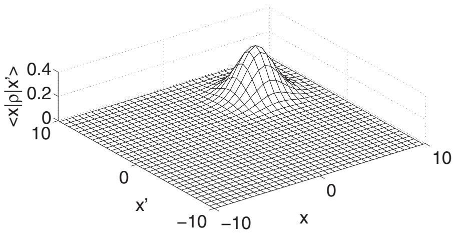

FIG. 6. Plot of $\langle x|\rho|x'\rangle$ for a coherent state $|\alpha\rangle$, where $\alpha=2.5$.

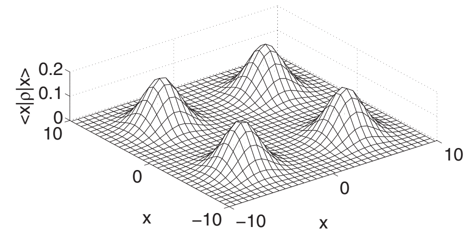

FIG. 7. Plot of $\langle x|\rho|x'\rangle$ for the superposition state (51), where $\alpha=2.5$.

volves at least the two extreme states with $|\alpha_0|^2=1.2$, but could include other coherent states with $|\alpha_0|<1.1$.

VIII. PREDICTIONS OF PARTICULAR QUANTUM STATES

We will now consider experimental tests of the inequalities derived above. An important point is that the criteria presented are sufficient to prove the existence of generalized macroscopic superpositions, but there are many macroscopic superpositions which do not satisfy the above criteria. Nevertheless there are some systems of current experimental interest which do allow for violation of the inequalities. We analyze such cases below, noting that the violation would be predicted without the experimenter needing to make assumptions about the particular state involved.

A. Coherent states

The wave function for the coherent state $|\alpha\rangle$ is

$$\langle x|\alpha\rangle = \frac{1}{(2\pi)^{1/4}}\exp\left\{\frac{-x^2}{4}+\alpha x-|\alpha|^2\right\}. \tag{47}$$

This gives the expansion in the continuous basis set $|x\rangle$, the eigenstates of $\hat{x}$. Thus for the coherent state

$$|\alpha\rangle = \sum_x c_x|x\rangle = \int \langle x|\alpha\rangle|x\rangle dx. \tag{48}$$

The probability distribution for x is the Gaussian (Fig. 4)

$$P(x) = |\langle x|\alpha\rangle|^2 = \frac{1}{(2\pi)^{1/2}}\exp\left\{\frac{-(x-2\alpha)^2}{2}\right\} \tag{49}$$

(we take α to be real) centered at 2α and with variance $\Delta^2 x=1$.

The coherent state possesses nonzero off-diagonal elements $\langle x|\rho|x'\rangle$ where $|x-x'|$ is large and thus strictly speaking can be regarded as a generalized macroscopic superposition. However, as x and x' deviate from 2α, the matrix elements decay rapidly, and the off-diagonal elements decay rapidly with increasing separation:

$$\langle x|\rho|x'\rangle = \frac{1}{(2\pi)^{1/2}}\exp\left\{\frac{-(x-2\alpha)^2}{4}+\frac{-(x'-2\alpha)^2}{4}\right\}. \tag{50}$$

In effect then, the off-diagonal elements become zero for significant separations $|x-x'|\geq 1$ (Fig. 6). We can expect that the detection of the macroscopic aspects of this superposition will be difficult. Since $\Delta p=1$, it follows that we can use the criterion (35) to prove coherence between outcomes of x separated by at most $S=2$ (Fig. 4), which corresponds to the separation $S=2\Delta x$.

B. Superpositions of coherent states

The superposition of two coherent states [47]

$$|\psi\rangle = (1/\sqrt{2})\{e^{i\pi/4}|-\alpha\rangle + e^{-i\pi/4}|\alpha\rangle\}, \tag{51}$$

where α is real and large is an example of a macroscopic superposition state. The wave function in the position basis is

$$\langle x|\psi\rangle = \frac{-ie^{i\pi/4}e^{[-x^2/4-\alpha^2]}}{\sqrt{2}(2\pi)^{1/4}}\{e^{\alpha x}+ie^{-\alpha x}\}.$$

We consider the two complementary observables $\hat{x}$ and $\hat{p}$, and note that the probability distribution $P(x)$ for $\hat{x}$ displays two Gaussian peaks centered on $x=\pm 2\alpha$ (Fig. 5): $P(x) = \frac{1}{2}\sum_{\pm}P_{G\pm}(x)$, where $P_{G\pm}(x)=\exp[-(x\mp 2\alpha)^2/2]/\sqrt{2\pi}$. Each Gaussian has variance $\Delta^2 x=1$.

The macroscopic nature of the superposition is reflected in the significant magnitude of the off-diagonal elements $\langle x|\rho|x'\rangle$, where $x=\pm 2\alpha$ and $x'=\mp 2\alpha$, corresponding to $|x-x'|=4\alpha$. In fact

$$|\langle x|\rho|x'\rangle| = \frac{e^{-(x^2+x'^2)/4-2\alpha^2}}{\sqrt{2\pi}}\sqrt{\cosh(2\alpha x)\cosh(2\alpha x')} \tag{52}$$

as plotted in Fig. 7 and which for these values of x and x' becomes $\frac{(1-e^{-8\alpha^2})}{2(2\pi)^{1/2}}$. With significant off-diagonal elements connecting macroscopically different values of x, this superposition is a good example of a generalized macroscopic superposition (7).

Nonetheless we show that the simple linear criteria (35) and (17) derived from Eq. (4) are not sufficiently sensitive to detect the extent of the macroscopic coherence of this superposition state (51), even though the state (51) cannot be written in the form (10). We point out that it may be possible to derive further nonlinear constraints from Eq. (10) to arrive at more sensitive criteria.

To investigate what can be inferred from criteria (35), we note that $\hat{x}$ is the macroscopic observable. The complementary observable $\hat{p}$ has distribution $P(p)=\exp[-p^2/2](1+\sin 2\alpha p)/\sqrt{2\pi}$ which exhibits fringes and has variance $\Delta^2 p=1-4\alpha^2\exp[-4\alpha^2]$ (Fig. 8). There is a maximum squeezing of $\Delta^2 p\approx 0.63$ at $\alpha=0.5$. However, the squeezing diminishes as α increases, so the criterion becomes less ef-

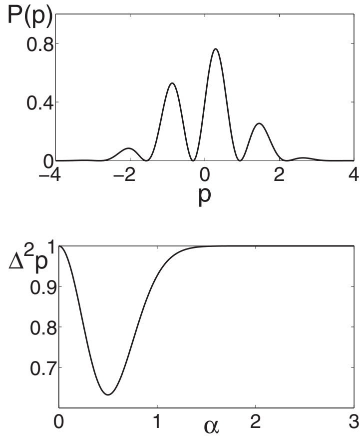

FIG. 8. (a) $P(p)$ for a superposition (51) of two coherent states where $\alpha=2.5$ and (b) the reduced variance $\Delta^2 p<1$, versus α.

fective as the separation of states of the macroscopic superposition increases. The maximum separation S that could be conclusively inferred from this criterion is $S\approx 2.5$ at $\alpha=0.5$.

As discussed in Sec. VII, the detection of squeezing in p is enough to confirm the system is not that of the mixture

$$\rho = 1/2(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|) \tag{53}$$

of the two coherent states. In fact, the squeezing rules out that the system is any mixture of coherent states. We note though that since the degree of squeezing Δp is small, our criteria is not sensitive enough to rule out superpositions of macroscopically separated coherent states.

C. Squeezed states

Consider the single-mode momentum squeezed state [48]

$$|\psi\rangle = e^{r(a^2-a^{\dagger 2})}|0\rangle. \tag{54}$$

Here $|0\rangle$ is the vacuum state. For large values of r these states are generalized macroscopic superpositions of the continuous set of eigenstates $|x\rangle$ of $\hat{x}=a+a^{\dagger}$, with wave function

$$\langle x|\psi\rangle = \frac{1}{(2\pi\sigma)^{1/4}}\exp\left\{\frac{-x^2}{4\sigma}\right\} \tag{55}$$

and associated Gaussian probability distribution

$$P(x) = \frac{1}{(2\pi\sigma)^{1/2}}\exp\left\{\frac{-x^2}{2\sigma}\right\}. \tag{56}$$

The variance is $\sigma=e^{2r}$. As the squeeze parameter r increases, the probability distribution expands, so that eventually with large enough r, x can be regarded as a macroscopic observable. This behavior is shown in Fig. 9. The distribution for p is also Gaussian but is squeezed, meaning that it has reduced variance: $\Delta^2 p<1$. In fact, Eq. (54) is a minimum uncertainty state, with $\Delta^2 p=1/\sigma=e^{-2r}$. Where squeezing is significant, the off-diagonal elements $\langle x|\rho|x'\rangle=\langle x|\psi\rangle\langle\psi|x'\rangle$ (where $|x-x'|$ is large) are significant over a large range of x values (Fig. 9).

The criterion (17) for the binned outcomes is violated for the ideal squeezed state (54) for values of S up to $0.5\sqrt{\sigma}$. The

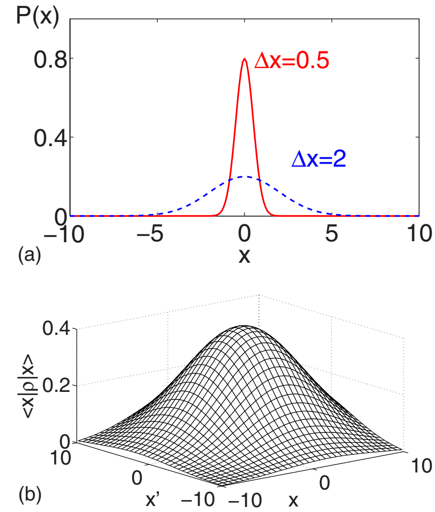

FIG. 9. (Color online) (a) Probability distribution for a measurement X for a momentum-squeezed state. The variance $\Delta^2 x$ increases with squeezing in p, to give a macroscopic range of outcomes, and for the minimum uncertainty state (54) satisfies $\Delta x\Delta p=1$. (b) The $\langle x|\rho|x'\rangle$ for a squeezed state (54) with $r=13.4$ ($\Delta x=3.67$ which predicts $\langle a^{\dagger}a\rangle=2.5^2$.

criterion can thus confirm macroscopic superpositions of states with separation of up to half the standard deviation of the probability distribution of x, even as $\Delta x\to\infty$. This behavior has been reported in [11] and is shown in Fig. 10.

Squeezed systems that are generated experimentally will not be describable as the pure squeezed state (54). This pure state is a minimum uncertainty state with $\Delta x\Delta p=1$. Typically experimental data will generate Gaussian probability distributions for both x and p and with squeezing $\Delta p<1$ in p, but typically $\Delta x\Delta p>1$. The maximum value of S that can be proved in this case of the Gaussian states reduces to 0 as

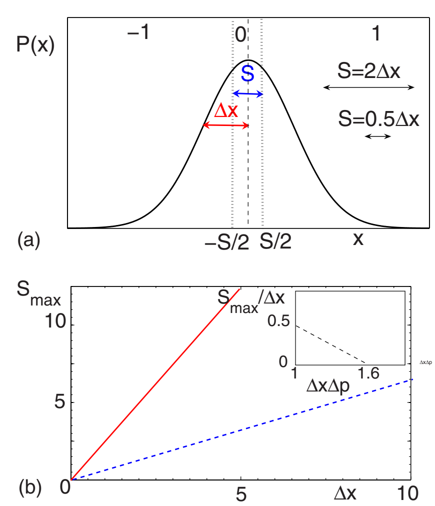

FIG. 10. (Color online) Detection of underlying superpositions of size S for the squeezed minimum uncertainty state (54) by violation of Eqs. (17) [dashed line of (b)] and (35) [full line of (b)]. S_{max} is the maximum S for which the inequalities are violated. Inset of (b) shows behavior of violation of Eq. (17) for general Gaussian-squeezed states. Inequality (35) depends only on Δp. The size of S_{max} relative to $P(x)$ is illustrated in (a).

$\Delta x\Delta p$ (or $\Delta x\Delta_{\mathrm{inf}}p$) increases to ~ 1.6. This is shown in Fig. 10. Analysis of recent experimental data for impure states that allows a violation of Eq. (17) has been reported by Marquardt *et al.* [12].

The criterion (35), as given by Theorem 4, is better able to detect the superpositions (Fig. 10), particularly where the uncertainty product gives $\Delta x\Delta p > 1$, though in this case the superpositions are nonlocatable in phase space, so that we cannot conclude an outcome domain for the states involved in the superposition. This criterion depends only on the squeezing Δp in one quadrature and is not sensitive to the product $\Delta x\Delta p$. For ideal squeezed states with variance $\Delta^2 x = \sigma$, one can prove a superposition of size $S=2\sqrt{\sigma}$, four times that obtained from Eq. (17) (Fig. 10). Experimental reports [29] of squeezing of orders $\Delta p \approx 0.4$ confirms superpositions of size at least $S=5$, which is 2.5 times that defined by $S=2$, which corresponds to two standard deviations of the coherent state, for which $\Delta x=1$ (Fig. 4).

D. Two-mode squeezed states

Next we consider the two-mode squeezed state [49]

$$e^{r(ab-a^\dagger b^\dagger)}|0\rangle|0\rangle. \tag{57}$$

Here a,b are boson annihilation operators for modes A and B, respectively. The wave function $\langle x|\psi\rangle$ and distribution $P(x)$ are as in Eqs. (55) and (56), but the variance in $\hat{x}=X^A$ is now given by $\sigma=\cosh 2r$. The $\hat{x}=X^A$ is thus a macroscopic observable.

In the two-mode case, the squeezing is in a linear combination P^A+P^B of the momenta P^A and P^B at A and B, rather than in the momentum $\hat{p}=P^A$ for A itself. The observable that is complementary to X^A is of form $\tilde{P}=P^A-gP^B$, where g is a constant, which is Eq. (19) of Sec. V. We can select to evaluate one of the criteria (23), (38), and (39).

Choosing as our macroscopic observable x and our complementary one P^A-gP^B, we calculate

$$\Delta^2_{\mathrm{inf}}p^A = 1/\sigma = 1/\cosh 2r \tag{58}$$

for the choice $g=\langle P^AP^B\rangle/\langle (P^B)^2\rangle = -\tanh r$ which minimizes $\Delta^2_{\mathrm{inf}}p^A$ [44]. The application of results to criterion (23) gives the result as in Fig. 10, to indicate detection of superpositions of size S where $S=0.5\sqrt{\sigma}$ for the ideal squeezed state (57), and the result shown in the inset of Fig. 10 if $\Delta x^A\Delta_{\mathrm{inf}}p^A > 1$.

The prediction for the criterion of Theorem 3, to detect superpositions in the position sum X^A+X^B by measurement of a narrowed variance in the momenta sum P^A+P^B, is also given by the results of Fig. 10. Calculation for the ideal state (57) predicts $\Delta^2(\frac{p^A+p^B}{\sqrt{2}})=e^{-2r}$ and $\Delta^2(\frac{x^A+x^B}{\sqrt{2}})=e^{+2r}$ which corresponds to that of the one-mode squeezed state. The prediction for the maximum value of S of Theorem 3 is therefore given by the dashed curves of Fig. 10, and the inset.

A better result is given by Eq. (38), if we are not concerned with the location of the superposition. Where we use Eq. (38), the degree of reduction in $\Delta^2_{\mathrm{inf}}p^A$ determines the size of superposition S that may be inferred. By Theorem 5, measurement of $\Delta_{\mathrm{inf}}p^A$ allows inference of superpositions of eigenstates of $\hat{x}$ separated by at least

$$S = 2/\Delta_{\mathrm{inf}}p^A. \tag{59}$$

Realistic states are not likely to be pure squeezed states as given by Eq. (57). Nonetheless the degree of squeezing indicates a size of superposition in X^A, as given by Theorem 5. Experimental values of $\Delta^2_{\mathrm{inf}}p^A \approx 0.76$ have been reported [22], to give confirmation of superpositions of size $S\approx 2.3$, which is 1.1 times the level of $S=2$ that corresponds to two standard deviations $\Delta x^A=1$ of the vacuum state (Fig. 4).

More frequently, it is the practice to measure squeezing in the direct sum P^A+P^B of momenta. The macroscopic observable is then the position sum X^A+X^B. The reports of measured experimental values indicate [23] $\Delta^2(\frac{p^A+p^B}{\sqrt{2}})\approx 0.4$, which according to Theorem 5 implies superpositions in $(X^A+X^B)/\sqrt{2}$ of size $S\approx 3.2$, of order 1.6 times the standard vacuum state level. The slightly better experimental result for the superpositions in the position sum may be understood since it has been shown by Bowen *et al.* [22] that, for the Gaussian squeezed states, the measurement of $\Delta^2_{\mathrm{inf}}p^A$ is more sensitive to loss than that of $\Delta^2(p^A+p^B)$. The $\Delta_{\mathrm{inf}}p^A$ is an asymmetric measure that enables demonstration of the EPR paradox [39,44], a strong form of quantum nonlocality [41,50].

IX. CONCLUSION

We have extended our previous work [11] and derived criteria sufficient to detect generalized macroscopic (or S-scopic) superpositions $(\Sigma_{k_1}^{k_2}c_k|x_k\rangle)$ of eigenstates of an observable $\hat{x}$. For these superpositions, the important quantity is the value S of the extent of the superposition, which is the range in prediction of the observable (S is the maximum of $|x_j-x_i|$ where $c_j,c_i\neq 0$). This quantity gives the extent of indeterminacy in the quantum prediction for $\hat{x}$. In this sense, there is a contrast with the prototype macroscopic superposition (of type $c_2|x_2\rangle+c_1|x_1\rangle$) that relates directly to the essay of Schrödinger [1]. Such a prototype superposition contains only the two states that have separation S in their outcomes for x. Nonetheless, we have discussed how the generalized superposition is relevant to testing the ideas of Schrödinger, in that such macroscopic superpositions are shown to be inconsistent with the hypothesis of a quantum system being in at most one of two macroscopically separated states.

We have also defined the concept of a generalized S-scopic coherence and the class of MUTs without direct reference to quantum mechanics. The former is introduced in Sec. IV as the assumption (3) and is associated to the failure of a generalized assumption of macroscopic reality. This assumption is that the system is in at most one of two macroscopically distinguishable states, but that these underlying states are not specified to be quantum states. The assumption of MUTs is that these component states do at least satisfy the quantum uncertainty relations. In the derivation of the criteria of this paper, only two assumptions are made: that the system does satisfy this generalized macroscopic (S-scopic) reality and that the theory is a MUT. These assumptions lead to inequalities, which, when violated, generate evidence that at least one of the assumptions must be incorrect.

We point out that if, in the event of violation of the inequalities, we opt to conclude the failure of the MUT assumption, then this does not imply quantum mechanics to be incorrect, but rather that it is incomplete, in the sense that the component states can themselves not be quantum states. It can be said then that violation of the inequalities of this paper implies at least one of the assumptions of generalized macroscopic (S-scopic) reality and the completeness of quantum mechanics is incorrect.

There is a similarity with the Einstein-Podolsky-Rosen argument [38]. In the EPR argument, the assumption of a form of realism (local realism) is shown to be inconsistent with the completeness of quantum mechanics. Therefore, as a conclusion of that argument, one is left to conclude that at least one of local realism and the completeness of QM is incorrect [39–41]. EPR opted for the first and took their argument as a demonstration that quantum mechanics was incomplete. Only after Bell [42] was it shown that this was an incorrect choice. Here, as in the EPR argument, the assumption of a form of realism [macroscopic (S-scopic) realism] can only be made consistent with the predictions of quantum mechanics if one allows a kind of theory in which the underlying states are not restricted by the uncertainty relations [11].

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APPENDIX A: PROOF OF THEOREM A

We will now prove the statement that coherence between x_1 and x_2 is equivalent to a nonzero off-diagonal element $\langle x_1|\rho|x_2\rangle$ in the density matrix. As discussed in Sec. II, within quantum mechanics the statement that there exists coherence between x_1 and x_2 is equivalent to the statement that there is no decomposition of the density matrix of form (2) where ρ_1 and ρ_2 are density matrices such that $\langle x_1|\rho_2|x_1\rangle=\langle x_2|\rho_1|x_2\rangle=0$. Therefore Theorem A can be reformulated as saying that $\langle x_1|\rho|x_2\rangle=0$ if and only if such a decomposition does exist.

It is easy to prove the first direction of the equivalence: if $\exists\{\wp_1,\wp_2,\rho_1,\rho_2\}$ such that $\rho=\wp_1\rho_1+\wp_2\rho_2$ and $\langle x_1|\rho_2|x_1\rangle=\langle x_2|\rho_1|x_2\rangle=0$, then $\langle x_1|\rho|x_2\rangle=0$. To show this, first note that for any density matrix $\bar{\rho}$ and $\forall\{x,x'\}$, if $\langle x|\bar{\rho}|x\rangle=0$ then $\langle x|\bar{\rho}|x'\rangle=0$, where $\langle x|x'\rangle=\delta_{x,x'}$. Since by assumption $\langle x_1|\rho_2|x_1\rangle=\langle x_2|\rho_1|x_2\rangle=0$, then $\langle x_1|\rho|x_2\rangle=\Sigma_i\wp_i\langle x_1|\rho_i|x_2\rangle=0$.

The converse can also be proved. We use the facts that any ρ can always be written as the reduced density matrix of an enlarged pure state, where the system of interest (call it A) is entangled with an ancilla B, i.e., $\rho=\mathrm{Tr}_B\{|\Psi_{AB}\rangle\langle\Psi_{AB}|\}$; and that any bipartite pure state can always be written in the Schmidt decomposition [51]

$$|\Psi_{AB}\rangle=\sum_i\sqrt{\eta_i}|\psi_i\rangle|\phi_i^B\rangle, \tag{A1}$$

where $\{|\psi_i\rangle\}$ and $\{|\phi_i^B\rangle\}$ are orthonormal and $\eta_i\in[0,1]$. The superscript B denotes the states of the ancilla and the absence of a superscript denotes the states of the system of interest A. We decompose each pure state $|\psi_i\rangle$ that appears in the Schmidt decomposition in the basis of eigenstates of $\hat{x}$ as $|\psi_i\rangle=\Sigma_k c_{i,k}|x_k\rangle$. By assumption $\langle x_1|\rho|x_2\rangle=0$ and therefore $\Sigma_i\eta_i\langle x_1|\psi_i\rangle\langle\psi_i|x_2\rangle=\Sigma_i\eta_i c_{i,1}c_{i,2}^*=0$. We can expand $|\Psi_{AB}\rangle$ as

$$|\Psi_{AB}\rangle=|x_1\rangle|\widetilde{1_B}\rangle+|x_2\rangle|\widetilde{2_B}\rangle+\sum_{k>2,i}\sqrt{\eta_i}c_{i,k}|x_k\rangle|\phi_i^B\rangle, \tag{A2}$$

where we define the (unnormalized) $|\widetilde{1_B}\rangle\equiv\Sigma_i\sqrt{\eta_i}c_{i,1}|\phi_i^B\rangle$ and $|\widetilde{2_B}\rangle\equiv\Sigma_i\sqrt{\eta_i}c_{i,2}|\phi_i^B\rangle$. The inner product of these two vectors is $\langle\widetilde{1_B}|\widetilde{2_B}\rangle=\Sigma_i\eta_i c_{i,1}c_{i,2}^*$. But as shown above $\Sigma_i\eta_i c_{i,1}c_{i,2}^*=0$, so $|\widetilde{1_B}\rangle$ and $|\widetilde{2_B}\rangle$ are orthogonal. We can therefore define an orthonormal basis with the (normalized) $|1_B\rangle=|\widetilde{1_B}\rangle/\sqrt{\Sigma_i\eta_i|c_{i,1}|^2}$ and $|2_B\rangle=|\widetilde{2_B}\rangle/\sqrt{\Sigma_i\eta_i|c_{i,2}|^2}$, plus additional $|j_B\rangle$ with $3\le j\le D$, where D is the dimension of subsystem B's Hilbert space. Taking the trace of $\rho_{AB}=|\Psi_{AB}\rangle\langle\Psi_{AB}|$ therefore yields

$$\rho=\mathrm{Tr}_B\{\rho_{AB}\}=\langle 1_B|\rho_{AB}|1_B\rangle+\langle 2_B|\rho_{AB}|2_B\rangle+\sum_{j>2}\langle j_B|\rho_{AB}|j_B\rangle. \tag{A3}$$

Now referring to expansion (A2), we see that $\langle 1_B|\rho_{AB}|1_B\rangle=\Sigma_i\eta_i|c_{i,1}|^2|x_1\rangle\langle x_1|$ and $\langle 2_B|\rho_{AB}|2_B\rangle=\Sigma_i\eta_i|c_{i,2}|^2|x_2\rangle\langle x_2|$. We then define $\rho_1\equiv|x_1\rangle\langle x_1|$, $\wp_1\equiv\Sigma_i\eta_i|c_{i,1}|^2$, $\wp_2=1-\wp_1$, and $\rho_2\equiv\frac{1}{\wp_2}\{\Sigma_i\eta_i|c_{i,2}|^2|x_2\rangle\langle x_2|+\Sigma_{j>2}\langle j_B|\rho_{AB}|j_B\rangle\}$. Obviously $\langle x_2|\rho_1|x_2\rangle=0$, and by substituting Eq. (A2) into ρ_2 we see that $\langle x_1|\rho_2|x_1\rangle=0$. Therefore ρ can be decomposed as $\rho=\wp_1\rho_1+\wp_2\rho_2$ with $\langle x_1|\rho_2|x_1\rangle=\langle x_2|\rho_1|x_2\rangle=0$ as desired.

APPENDIX B

We wish to prove that if ρ can be written as $\rho_{\mathrm{mix}}=\wp_L\rho_L+\wp_R\rho_R$, then $\Delta^2_{\mathrm{inf,mix}}p^A\ge\wp_L\Delta^2_{\mathrm{inf},L}p^A+\wp_R\Delta^2_{\mathrm{inf},R}p^A$, where

$$\Delta^2_{\mathrm{inf},J}p^A=\sum_{p^B}\wp_J(p^B)\Delta^2_J(p^A|p^B).$$

The subscript J refers to the ρ_J from which the probabilities are calculated.

We have

$$\begin{aligned}\Delta^2_{\mathrm{inf,mix}}p^A&=\sum_{p^B}P_{\mathrm{mix}}(p^B)\Delta^2_{\mathrm{mix}}(p^A|p^B)\\&=\sum_{p^B}\sum_{p^A}P_{\mathrm{mix}}(p^A,p^B)(p^A-\langle p^A|p^B\rangle_{\mathrm{mix}})^2\\&\ge\sum_{p^B}\sum_{p^A}\sum_{I=R,L}\wp_I P_I(p^A,p^B)(p^A-\langle p^A|p^B\rangle_I)^2.\end{aligned}$$

The inequality follows because $\langle p^A|p^B\rangle_{\mathrm{mix}}$ is the mean of $P(p^A|p^B)$ for ρ_{mix}, and the choice $a=\Sigma_p P(p)p=\langle p\rangle$ will minimize $\Sigma_p P(p)(p-a)^2$. From this the required result follows.

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