Dual symmetric Lagrangians in quantum electrodynamics I: Conservation laws and multipolar coupling

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Abstract

By using a complex field with a symmetric combination of electric and magnetic fields, a first-order covariant Lagrangian for Maxwell's equations is obtained, similar to the Lagrangian for the Dirac equation. This leads to a dual-symmetric quantum electrodynamic theory with an infinite set of local conservation laws. The dual symmetry is shown to correspond to a helical phase, conjugate to the conserved helicity. There is also a scaling symmetry, conjugate to the conserved entanglement. The results include a novel form of the photonic wave-function, with a well-defined helicity number operator conjugate to the chiral phase, related to the fundamental dual symmetry. Interactions with charged particles can also be included. Transformations from minimal-coupling to multi-polar or more general forms of coupling are particularly straightforward using this technique. The dual-symmetric version of quantum electrodynamics derived here has potential applications to nonlinear quantum optics and cavity quantum electrodynamics.

I. INTRODUCTION

The complex form of Maxwell's equations[1] using $\mathbf{E} + i\mathbf{B}$ has been known nearly since the invention of electrodynamics itself. The complex phase introduced in these equations transforms electric to magnetic fields, and thus generates the electric-magnetic symmetry property known as electromagnetic dual symmetry[2]. This is related to a larger class of symmetries found elsewhere in quantum field theory[3]. In an earlier paper[4], a suitable Lagrangian was introduced using a complex gauge field, for this form of Maxwell's equations. This resulted in a Dirac type of Lagrangian[5] for electrodynamics, with only first-order time-derivatives. The analogy with the Dirac equation for spin one-half particles extends to a covariant Lagrangian, involving a six-dimensional complex field. The dual symmetry (interchanging $\mathbf{B}$ and $\mathbf{E}$) is generated simply through a phase rotation of the complex field. It is related to the conserved helicity number of the field - just as in Weyl[6] neutrino theory. The dual phase is conjugate to a photon-number difference operator, so it has a corresponding hermitian phase operator[7, 8].

It is also possible to develop a local current and density operator for the helicity number, which does not display the known problems found with attempts to construct a local photon number density[9, 10]. This has the advantage over some previous proposals that it is related to physical fields acting at a point, has well-defined transformation properties, and gives information about particle number rather than energy. The helicity density operator effectively takes advantage of the explicit dual-symmetry in the present formulation of electrodynamics. While this quantity could be calculated from traditional approaches, it is more complicated to derive in the usual electrodynamics. Another conserved current is obtained from a rescaling symmetry, which is related to squeezing and entanglement.

These properties suggest that dual symmetry in electrodynamics has fundamental significance. It provides a transparent route towards understanding the conserved currents and internal symmetries that are present in a free electromagnetic field. There are an infinite number of conservation laws, including the traditional space-time symmetries like energy and momentum conservation. These are readily derived from the complex Maxwell equations. Some of these are derivable from the symmetries of the free Lagrangian. Clearly, obtaining conservation laws from symmetry principles provides additional insight.

The present paper gives a detailed account of these conservation laws. Relativistic trans-

formations and Lorenz symmetry are also treated. The dual symmetric Lagrangian approach is then extended to include interactions with charged particles. Either minimal[5] or multi-polar[11–13] couplings are obtainable with the first-order Lagrangian. The canonical transformation to a multi-polar form is transparent, since it is achieved through a simple variable change in the complex dual potential. Thus, there is no difficulty in regaining standard results for interactions of charged particles with the radiation field.

Since the dual-symmetric technique provides an especially clear transformation to the multi-polar form of the Hamiltonian[11, 12], these methods can readily handle interactions with bound states in atoms or molecules. In the multipolar formulation, the relevant dual symmetry links the displacement and magnetic fields. The Lagrangian can be easily extended to include macroscopic dielectric and magnetic media with either linear or nonlinear response properties. There is an advantage in investigating hybrid or more general canonical couplings[14] with this form of electrodynamics.

In summary, this paper will focus on fundamental issues: conservation laws, interactions with charged particles, the quantization of macroscopic electric and magnetic media, and multipolar transformations. As an immediately useful application, the results will be employed in a subsequent paper to treat the interaction of atoms with dielectric or magnetic environments.

II. WAVE-EQUATIONS

Bialynicki-Birula[21] has suggested the photonic wave-function (for a single photon) can be expressed in the form of the complex field [1] $\mathbf{E} + i\mathbf{B}$, where a simplified notation is used with units so that $\hbar = c = \varepsilon_o = \mu_o = 1$. The motivation for this proposal is from the Dirac equation. By using a complex field, it is possible to extend Dirac's original argument[5], to obtain a wave-equation for a massless spin-1 rather than spin-1/2 particle. This is similar to Dirac's technique of finding the square-root of the Laplacian operator, except using representations of O(3) rather than the Dirac gamma matrices. This technique has been used previously[15–20] and has been investigated recently as a means of defining a photonic wave-function[21–23]. The argument leading to a complex field wave-equation for Maxwell's equation will be summarized here, although with a different interpretation to those used earlier.

A. General wave-equation

Following Bialynicki-Birula[23], taking a matrix square-root of *any* second-order vector wave-equation

$$\frac{\partial^2}{\partial t^2}\Psi = \nabla^2\Psi \ , \tag{2.1}$$

together with the assumption that $\nabla \cdot \Psi = 0$, leads to a first-order equation of form:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi &= \pm i\left[\nabla \cdot \mathbf{S}\right]\Psi \\ &= \mp \nabla \times \Psi \ . \end{aligned} \tag{2.2}$$

Here $\mathbf{S}$ is a vector of 3×3 complex matrices such that $[S^i, S^j] = i\varepsilon_{ijk}S^k$, so that the $\mathbf{S}$ matrices can be represented as 3×3 hermitian rotation matrices:

$$\begin{aligned} \mathbf{S} &= (S^1, S^2, S^3) \\ &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) . \end{aligned} \tag{2.3}$$

The relation $\left[\nabla \cdot \mathbf{S}\right]\Psi = i\nabla \times \Psi$ is used to obtain the wave-equation above.

B. Photonic wave-function

To proceed further, it is necessary to relate the complex 'photonic wave-function' Ψ obtained from the Dirac square-root procedure, to physical fields. Some candidates for this can be readily determined by examining the complex form of Maxwell's equations, which have been known since the earliest days of electromagnetic theory[1].

The free-space form of Maxwell's equations can be written as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \ , \\ \dot{\mathbf{E}} &= \nabla \times \mathbf{B} \\ \dot{\mathbf{B}} &= -\nabla \times \mathbf{E} \ . \end{aligned} \tag{2.4}$$

Introducing

$$\boldsymbol{\mathcal{E}} = \mathbf{E} + i\mathbf{B} \ , \tag{2.5}$$

the complex field equations equivalent to (2.4) are:

$$\begin{aligned} \nabla \cdot \boldsymbol{\mathcal{E}} &= 0 \\ i\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} &= \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} \ . \end{aligned} \tag{2.6}$$

Equation (2.6) above has the same general form as the quantum wave-equation (2.2). However, $\boldsymbol{\mathcal{E}}$ is not the only candidate for a photonic wave-function. A potentially more useful and interesting possibility is found by considering Lorenz-invariance and symmetries, which will be treated in the next section.

Irrespective of these questions, the complex form of Maxwell equations is a very clear and simple way to express the electromagnetic wave-equation as a single, first-order equation. It seems to be rather easily applicable to many problems in modern optics where exact solutions beyond the paraxial approximation are needed, as in the use of complex focal methods[24] to obtain focused solutions at high Fresnel number.

C. Dual potential

The complex field equation for $\boldsymbol{\mathcal{E}}$ is a useful form in the case of a single photon, but was not derived from a Lagrangian. This is necessary to have a clear action principle or path integrals, for field quantization. In order to extend the dual symmetric approach to a Lagrangian, which is essential for canonical quantization, it is useful to define a dual potential field. This is carried out by introducing a transverse complex vector potential

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\Lambda} + i\mathbf{A} \, , \tag{2.7}$$

where a generalized Coulomb gauge is chosen so that $\nabla \cdot \boldsymbol{\mathcal{A}} = 0$. The field $\boldsymbol{\Lambda}$ is the dual potential for the transverse part of the electric field, while $\mathbf{A}$ is the usual magnetic vector potential. Both magnetic and electric components appear in the complex potential field $\boldsymbol{\mathcal{A}}$, which is related to the free complex Maxwell field $\boldsymbol{\mathcal{E}}$ by:

$$\boldsymbol{\mathcal{E}} = \nabla \times \boldsymbol{\mathcal{A}} \, . \tag{2.8}$$

In general $\mathbf{E}$ can have a longitudinal component, which has no representation in this form. This problem only arises in the presence of interactions. Interacting fields will be treated in more detail in a later section, and this extension is especially straightforward in

a multipolar gauge approach. It is instructive to proceed for the moment in the case of free fields, without initially taking account of longitudinal fields whose source is in the free charges.

D. Dual symmetric Lagrangian

An essential question for quantization of the complex field equations, is whether there is a Lagrangian that can generate Eq. (2.6), while having a Hamiltonian density equal to the free energy density

$$\mathcal{H}_0\left(\boldsymbol{\mathcal{E}}\right) = \frac{1}{2}\boldsymbol{\mathcal{E}}^* \cdot \boldsymbol{\mathcal{E}} = \frac{1}{2}\left[|\mathbf{E}|^2 + |\mathbf{B}|^2\right] \ . \tag{2.9}$$

In units where $\hbar = c = 1$, energy has dimension inverse to time, and hence also to space. The Lagrangian density $\mathcal{L}$ has the same dimension as the energy density $\mathcal{H}$, which must be $[\ell]^{-4}$, where $[\ell]$ is a length unit. From the energy density equation (2.9), it is clear that the complex field $\boldsymbol{\mathcal{E}}$ has dimension $[\ell]^{-2}$. Using (2.8), the complex field is a spatial derivative of the complex potential $\boldsymbol{\mathcal{A}}$, which therefore has dimension $[\ell]^{-1}$.

From these dimensional results, any quadratic Lagrangian requires either a combination of two potentials with two space-time derivatives, or else a product of field and potential with a single derivative. Both routes are possible and equivalent. There is a Lagrangian density just involving the complex potentials with these properties[4]. It can be written as:

$$\mathcal{L}_0\left(\boldsymbol{\mathcal{A}}, \dot{\boldsymbol{\mathcal{A}}}\right) = \frac{1}{2}\left[i\dot{\boldsymbol{\mathcal{A}}} \cdot \nabla \times \boldsymbol{\mathcal{A}}^* - |\nabla \times \boldsymbol{\mathcal{A}}|^2\right] \ . \tag{2.10}$$

It is simple to verify that this generates the field equation, using either ordinary or transverse variational calculus, since:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}_0}{\partial \dot{\boldsymbol{\mathcal{A}}}} = \frac{i}{2}\nabla \times \dot{\boldsymbol{\mathcal{A}}}^* = -\frac{1}{2}\nabla \times \left(\nabla \times \boldsymbol{\mathcal{A}}^*\right) \ . \tag{2.11}$$

This equation is identical to the complex field equation (2.6), and hence to the usual free field Maxwell equations. The canonical momentum conjugate to $\boldsymbol{\mathcal{A}}$ is

$$\boldsymbol{\Pi} = \frac{i}{2}\boldsymbol{\mathcal{E}}^* \ . \tag{2.12}$$

Provided there are no charges present, the resulting Hamiltonian density is identical to the energy density:

$$\begin{aligned} \mathcal{H}_0\left(\boldsymbol{\mathcal{E}}\right) &= \boldsymbol{\Pi} \cdot \dot{\boldsymbol{\mathcal{A}}} - \mathcal{L}_0 \\ &= \frac{1}{2}\left|\nabla \times \boldsymbol{\mathcal{A}}\right|^2 = \frac{1}{2}\boldsymbol{\mathcal{E}}^* \cdot \boldsymbol{\mathcal{E}} \, . \end{aligned} \tag{2.13}$$

In summary, the wave-equation for a photon is easily extended to a Lagrangian field theory, if the complex vector field $\boldsymbol{\mathcal{E}}$ is obtained from a complex gauge field $\boldsymbol{\mathcal{A}}$. From Eq (2.11), together with the transverse gauge constraint, the time-evolution equation for $\boldsymbol{\mathcal{A}}$ is formally identical to the complex field equation:

$$i\frac{\partial \boldsymbol{\mathcal{A}}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}} \, . \tag{2.14}$$

III. QUANTUM FIELD THEORY

While commutators and other properties of the quantized complex electromagnetic field can be worked out more traditionally[25, 26], it is most natural to quantize the complex field directly using the Lagrangian. This has the great advantage of making the symmetries, conservation laws and multipolar transformations particularly transparent. It is also easily generalized to treat interacting fields.

A. Quantization

Having introduced the Lagrangian, this can be quantized by either the constrained, transverse field, path-integral technique, or else by creating commutators from the fundamental constrained Dirac brackets[27], using functional integration over transverse fields.

From Eq (2.12), the canonical momentum conjugate to $\boldsymbol{\mathcal{A}}$ interchanges the magnetic and electric fields, giving:

$$\widehat{\boldsymbol{\Pi}} = \frac{1}{2}\left(\widehat{\mathbf{B}} + i\widehat{\mathbf{E}}\right) \, . \tag{3.1}$$

Imposing canonical commutators at equal times, and assuming transverse field constraints, gives:

$$\left[\widehat{\mathcal{A}}_i\left(t,\mathbf{x}\right), \widehat{\Pi}_j\left(t,\mathbf{x}'\right)\right] = i\delta^{\perp}_{ij}\left(\mathbf{x}-\mathbf{x}'\right) \, . \tag{3.2}$$

Here $\delta^{\perp}_{ij}\left(\mathbf{x}-\mathbf{x}'\right)$ is the transverse delta-function. The resulting field commutators agree with those derived in more standard ways, and hence agree with known quantum field theoretic results. Related dual symmetric Lagrangians of a different type were introduced by Zwanziger[28], and by Schwarz and Sen[29], using pairs of real fields, rather than the present approach.

Since $\left[\widehat{\mathcal{A}}_i^* (t,\mathbf{x}), \widehat{\Pi}_j (t,\mathbf{x}')\right] = 0$, as these are regarded to be independent canonical coordinates, this implies the usual commutators for the (transverse) real fields, namely:

$$\begin{aligned} \left[\widehat{\Lambda}_i (t,\mathbf{x}), \widehat{B}_j (t,\mathbf{x}')\right] &= i\delta_{ij}^{\perp} (\mathbf{x}-\mathbf{x}') \\ \left[\widehat{E}_i^{\perp} (t,\mathbf{x}), \widehat{A}_j (t,\mathbf{x}')\right] &= i\delta_{ij}^{\perp} (\mathbf{x}-\mathbf{x}') \ . \end{aligned} \tag{3.3}$$

As $\widehat{\mathbf{E}}^{\perp} = \nabla\times\widehat{\mathbf{\Lambda}}$, and $\widehat{\mathbf{B}} = \nabla\times\widehat{\mathbf{A}}$, the two commutators are consistent with each other, because $\delta_{ij}^{\perp} (\mathbf{x}-\mathbf{x}') \equiv \delta_{ji}^{\perp} (\mathbf{x}'-\mathbf{x})$. The important point is that the dual symmetric Lagrangian does reproduce these standard commutators, and hence agrees with known quantum field theoretic results.

While the usual prescription of Dirac canonical quantization was followed here, it is known that not all Lagrangian approaches to quantization are equivalent. The present results show that the dual symmetric Lagrangian approach - at least at the free-field level - is entirely equivalent to the usual Fermi Lagrangian approach, which is important in view of the high accuracy of perturbation theory calculations in QED. There are conceptual advantages in having a more transparent electric-magnetic symmetry. However, the chief merit of the method is that certain calculations are made more directly and simply with dual symmetric Lagrangians.

B. Mode expansion

In order to simplify calculations of conserved quantities, the fields can be expanded in terms of photonic annihilation and creation operators. The resulting mode expansion of the complex field has unusual properties. On expanding in terms of annihilation and creation operators for the different helicities, the complete mode expansion can be most simply written using positive and negative helicity fields $\widehat{\boldsymbol{\mathcal{A}}}_\sigma$ such that $\widehat{\boldsymbol{\mathcal{A}}} (\mathbf{x}) = \widehat{\boldsymbol{\mathcal{A}}}_+ (\mathbf{x}) + \widehat{\boldsymbol{\mathcal{A}}}_- (\mathbf{x})$, which are expanded as:

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{A}}}_+ (\mathbf{x}) &= \int d^3\mathbf{k}\, \mathbf{u}_{\mathbf{k}+} (\mathbf{x})\, \widehat{a}_{\mathbf{k}+} \\ \widehat{\boldsymbol{\mathcal{A}}}_- (\mathbf{x}) &= \int d^3\mathbf{k}\, \mathbf{u}_{\mathbf{k}-}^* (\mathbf{x})\, \widehat{a}_{\mathbf{k}-}^{\dagger}. \end{aligned} \tag{3.4}$$

Here $\widehat{a}_{\mathbf{k}\sigma}$, $\widehat{a}_{\mathbf{k}\sigma}^{\dagger}$ are creation and annihilation operators respectively, defined with the usual commutators:

$$\left[\widehat{a}_{\mathbf{k}\sigma}, \widehat{a}_{\mathbf{k}'\sigma'}^{\dagger}\right] = \delta^3 (\mathbf{k}-\mathbf{k}')\, \delta_{\sigma\sigma'} \ . \tag{3.5}$$

In order to satisfy the full vector field equation, (2.14), each complex helicity field must satisfy the complex Maxwell equation:

$$i\frac{\partial \boldsymbol{\mathcal{A}}_\sigma}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_\sigma \ . \tag{3.6}$$

Noting that in the Heisenberg picture $\widehat{a}_{\mathbf{k}\sigma}(t) \propto e^{-i\omega_k t}$, this implies that the mode functions $\mathbf{u}_{\mathbf{k}\sigma}$ are defined so that

$$k\mathbf{u}_{\mathbf{k}\sigma} = \sigma\nabla \times \mathbf{u}_{\mathbf{k}\sigma} \ , \tag{3.7}$$

where $\sigma = \pm 1$ and $k = \omega_k = |\mathbf{k}|$.

In other words, the complex field contains annihilation operators for positive helicity photons, and creation operators for negative helicity photons. From this perspective, the expansion given above is also an expansion of the complex field into positive and negative frequency components, which is an approach used in photo-detection theory[44].

The normalization of $\mathbf{u}_{\mathbf{k}\sigma}$ is obtained from the requirement that the integrated Hamiltonian energy density of Eq (2.9) matches the photon energy. Since the quantum-classical correspondence requires a definite ordering to be chosen, it is usual to choose normal ordering (indicated here by $:\cdots:$), with all the creation operators to the left:

$$\begin{aligned} \widehat{H}_0 &= \frac{1}{2}\int :\widehat{\boldsymbol{\mathcal{E}}}^\dagger(\mathbf{x})\,\widehat{\boldsymbol{\mathcal{E}}}(\mathbf{x}): d^3\mathbf{x} \\ &= \sum_\sigma \int d^3\mathbf{k}\omega_{\mathbf{k}} \widehat{a}^\dagger_{\mathbf{k}\sigma}\widehat{a}_{\mathbf{k}\sigma} \ . \end{aligned} \tag{3.8}$$

This defines the normalization of the mode functions, such that $|\mathbf{k}\times\mathbf{u}_{\mathbf{k}\sigma}|^2 = |k\mathbf{u}_{\mathbf{k}\sigma}|^2 = k/(4\pi^3)$. Using periodic boundary conditions, $\mathbf{u}_{\mathbf{k}\pm}(\mathbf{x})$ corresponds to positive and negative helicity photons respectively, with:

$$\mathbf{u}_{\mathbf{k}\sigma}(\mathbf{x}) = \frac{1}{\sqrt{4\pi^3 k}} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k}\sigma} \ , \tag{3.9}$$

provided that $\mathbf{k}\cdot\mathbf{e}_{\mathbf{k}\sigma} = \mathbf{e}_{\mathbf{k}\sigma}\cdot\mathbf{e}_{\mathbf{k}\sigma} = 0$, with $i\mathbf{k}\times\mathbf{e}_{\mathbf{k}\sigma} = \sigma k\mathbf{e}_{\mathbf{k}\sigma}$, $\mathbf{e}_{\mathbf{k}\sigma}\cdot\mathbf{e}^*_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}$, and $\mathbf{e}_{\mathbf{k}\sigma} = \mathbf{e}^*_{\mathbf{k}-\sigma}$. In standard optical terminology, $\sigma = 1$ implies 'left-handed' circular polarization, and $\sigma = -1$ 'right-handed' circular polarization[26, 30].

Using plane-waves with periodic boundary conditions, the complete mode expansion then becomes:

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{A}}}(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{\sqrt{4\pi^3 k}}\mathbf{e}_{\mathbf{k}+}\left[\widehat{a}_{\mathbf{k}+}e^{i\mathbf{k}\cdot\mathbf{x}} + \widehat{a}^\dagger_{\mathbf{k}-}e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ \widehat{\boldsymbol{\mathcal{E}}}(\mathbf{x}) &= \int \frac{k d^3\mathbf{k}}{\sqrt{4\pi^3 k}}\mathbf{e}_{\mathbf{k}+}\left[\widehat{a}_{\mathbf{k}+}e^{i\mathbf{k}\cdot\mathbf{x}} - \widehat{a}^\dagger_{\mathbf{k}-}e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \end{aligned} \tag{3.10}$$

These results also agree with those obtained using conventional quantization methods[31].

C. Covariant wave-equation

An alternative quantization procedure is to use a covariant local Lagrangian that has only first-order derivatives, so that it directly generates the first-order Maxwell equations, just as the Dirac Lagrangian directly generates the Dirac equation[5]. This has the advantage of putting photon and particle Lagrangians on a symmetric footing, with both types of quantum fields obtained from quantizing similar first-order Lagrangians. It also leads to better understanding of symmetries.

Before considering the construction of a covariant Lagrangian, it is useful to recall that the original Dirac argument provided the fundamental spin-half wave-equation in a covariant form in which space and time enter in an equivalent way. In the present case, the first order Maxwell equations can be written in a way that is formally identical to the Dirac equation, as:

$$i\partial_\mu \alpha^\mu \Psi = p_\mu \alpha^\mu \Psi = 0 \ , \tag{3.11}$$

where $p_\mu = i\partial_\mu = i\partial/\partial x^\mu$ is the usual momentum operator of a quantum wave-equation. The α^μ matrices have a number of representations, one being just the previous one in Eq (2.2), that $\alpha^0 = S^0 = I$ and $\alpha^i = S^i$. In this case one must choose either $\Psi = \boldsymbol{\mathcal{E}}$ or $\Psi = \boldsymbol{\mathcal{A}}$. There is no obvious Lagrangian that can be formed just using the field $\boldsymbol{\mathcal{E}}$, for dimensional reasons. The dimension of $\boldsymbol{\mathcal{E}}$ is $[\ell]^{-2}$, while a Lagrangian density has to have dimension $[\ell]^{-4}$. Thus, a combination of quadratic terms in $\boldsymbol{\mathcal{E}}$ together with one derivative operator, would have dimension of $[\ell]^{-5}$. One must rule out the choice of $\Psi = \boldsymbol{\mathcal{A}}$ on similar grounds, as this would give a dimension of $[\ell]^{-3}$.

It is possible to extend the present complex electrodynamic field to a six-dimensional field vector, which does allow a construction of a Dirac-type first-order Lagrangian, on defining:

$$\Psi = \frac{1}{2} \begin{bmatrix} \boldsymbol{\mathcal{E}} \\ \boldsymbol{\mathcal{A}} \end{bmatrix} \ . \tag{3.12}$$

A larger 6×6 complex representation of the equation of motion is obtained on introducing matrices α^μ given by:

$$\alpha^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \ , \ \ \alpha^i = \begin{bmatrix} 0 & S^i \\ S^i & 0 \end{bmatrix} \ . \tag{3.13}$$

The wave-function Ψ given in Eq (3.12) generates the wave-equation (3.11), and allows for a covariant Lagrangian with correct units to be constructed as well, as shown next. Another possibility is to define a six-dimensional complex field in which $\boldsymbol{\mathcal{E}}$ appears twice[21, 23], but this has the drawback that there is no corresponding Lagrangian.

Although the term vector is used here, some caution is necessary: the term vector is defined purely in its mathematical sense as an ordered list of components. The actual transformation laws under a change of reference frame are nonlocal, and hence more complex than commonly used kinematic vector fields considered in physics. The transformations are considered and described in detail in a later section.

D. Covariant Lagrangian

The form of Lagrangian given in Eq (2.10) above is not explicitly Lorentz-covariant, and involves products of derivatives - rather than being just linear in the field derivatives, as in the Dirac Lagrangian. However, there is a well-known covariant form of the Lagrangian due to Schwinger[32], which does indeed have only first-order derivatives. In fact, it is known that many forms of the Lagrangian for electrodynamics are possible[33].

A covariant, first-order Lagrangian like the Dirac Lagrangian, (and different from the Schwinger Lagrangian) is given by combining products of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{A}}$, together with single first-order derivatives in space or time:

$$\begin{aligned} \mathcal{L}_C &= \Psi^\dagger p_\mu \alpha^\mu \Psi \\ &= \frac{1}{4}[\boldsymbol{\mathcal{A}}^* \cdot (i\dot{\boldsymbol{\mathcal{E}}} - \nabla \times \boldsymbol{\mathcal{E}}) + \boldsymbol{\mathcal{E}}^* \cdot (i\dot{\boldsymbol{\mathcal{A}}} - \nabla \times \boldsymbol{\mathcal{A}})] \ . \end{aligned} \tag{3.14}$$

Here $p_\mu = i\partial_\mu = i\partial/\partial x^\mu$ in standard four-vector notation where $x = (t, \boldsymbol{x})$, the indices $\mu = 0, \ldots 3$, are summed using the Einstein summation convention, and the α^μ matrices are the 6×6 matrices given in Eq (3.13).

This Lagrangian density generates the covariant form of the wave-equation given by Eq.(3.11). In more detail, the two three-vector components each satisfy an independent wave-equation of form:

$$\begin{aligned} i\dot{\boldsymbol{\mathcal{E}}} &= \nabla \times \boldsymbol{\mathcal{E}} \\ i\dot{\boldsymbol{\mathcal{A}}} &= \nabla \times \boldsymbol{\mathcal{A}} \ . \end{aligned} \tag{3.15}$$

The generation of the correct dynamical equations is not enough to guarantee that the Lagrangian is appropriate for quantization, unless the canonical Hamiltonian corresponds to the classical energy. This is satisfied, provide the initial condition is imposed that $\boldsymbol{\mathcal{E}} = \nabla \times \boldsymbol{\mathcal{A}}$. The Hamiltonian then equals the energy at all times. This follows since the canonical momentum field is

$$\Pi_\Psi = i\Psi^\dagger \ . \tag{3.16}$$

After integrating by parts and neglecting boundary terms, (provided $\Psi \to 0$ as $|\mathbf{r}| \to \infty$), the Hamiltonian density is clearly Hermitian, and has the usual form:

$$\begin{aligned} \mathcal{H}_C &= -\Psi^\dagger \mathbf{p} \cdot \boldsymbol{\alpha} \Psi \\ &= \frac{1}{2}\left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right) \ . \end{aligned} \tag{3.17}$$

After quantizing, the first-order covariant Lagrangian for Maxwell's equations has an almost identical pattern to that for the Weyl[6, 25] theory of the massless neutrino. One difference is that it is not possible to reduce the number of components below six in this case. Due to the dimensional restrictions in obtaining a result with units $[\ell]^{-4}$, any Dirac-type combination of quadratic terms together with one derivative, requires both the electromagnetic fields and their potentials. In other words, the fundamental fields are regarded here as being both the electromagnetic fields and their potentials, rather than the usual situation of just the potentials being treated as dynamical variables.

This does not change the number of physical degrees of freedom. Firstly, the use of complex vector constraints ($\boldsymbol{\mathcal{E}} = \nabla \times \boldsymbol{\mathcal{A}}$; $\nabla \cdot \boldsymbol{\mathcal{A}} = 0$) means that four complex degrees of freedom are removed due to constraints. This leaves two complex degrees of freedom in the first-order Lagrangian, just as in two-component (massless) Weyl neutrino theory. Secondly, these two complex degrees of freedom in a first-order Lagrangian are identical to the two real degrees of freedom of a second-order Lagrangian, corresponding operationally to the two helicity components.

An advantage of this approach is the physical appeal of having locally measurable fields appearing in the Lagrangian, rather than having a Lagrangian with only potentials present, as in most previous approaches.

E. Relativistic Transformations

The price that is paid for the explicit Lorenz-covariance obtained above is that the vector field Ψ introduced here has nonlocal transformation properties under Lorenz transformations. In fact, spinor representations for the electric and magnetic fields are well-known, and correspond to a finite-dimensional representation of the Lorenz group. If the four-dimensional coordinate Lorenz transformation is $x = \mathbf{L}\tilde{x}$, the transformation law for a boost transformation in the $\mathbf{v}$ direction (written explicitly in vector notation for clarity), is:

$$\begin{aligned} \widetilde{\boldsymbol{\mathcal{E}}}_{\parallel}(\tilde{x}) &= \boldsymbol{\mathcal{E}}_{\parallel}(x) \\ \widetilde{\boldsymbol{\mathcal{E}}}_{\perp}(\tilde{x}) &= \gamma[\boldsymbol{\mathcal{E}}_{\perp}(x) - i\mathbf{v} \times \boldsymbol{\mathcal{E}}_{\perp}(x)] \,. \end{aligned} \tag{3.18}$$

This transformation preserves the transverse nature of the complex electromagnetic field in the new frame of reference, and is completely local. However, the transformation law for the complex components of ψ corresponding to the potential $\boldsymbol{\mathcal{A}}$ are generally nonlocal due to the gauge choice imposed here, as is generally the case with the use of the Coulomb gauge.

The transverse potentials transform non-locally as:

$$\widetilde{\boldsymbol{\mathcal{A}}}(\tilde{y}) = \int d^3\tilde{\mathbf{x}} \boldsymbol{\delta}^{\perp}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}) \cdot \left(\gamma \boldsymbol{\mathcal{A}}_{\parallel}(\mathbf{L}\tilde{x}) + \boldsymbol{\mathcal{A}}_{\perp}(\mathbf{L}\tilde{x})\right) \,, \tag{3.19}$$

where $\boldsymbol{\delta}^{\perp}$ is the tensor representing the transverse delta function which projects the potentials onto their transverse components in the new frame, in order to preserve the transversality constraint. Because the constrained potentials have a one-to-one relationship with the transverse fields, it is clear that this transformation law also represents the Lorenz group. In a formal sense, the potential transformation belongs to an infinite-dimensional representation of the Lorenz group[34], since the Lorenz-transformed potentials must depend non-locally on the potentials in the original reference frame.

This is a consequence of the known difficulties associated with defining a unique photon position. It does not however, lead to any causality issues[35]. The nonlocality of the transformation only occurs in a plane orthogonal to any photon propagation direction. It therefore has no effect for photons that are spatially localized parallel to their propagation direction in one frame, and observed in another frame with relative velocity also parallel to the photon propagation direction. Since there is no state in which a photon is localized in all three dimensions simultaneously, the nonlocality of the Lorenz transformation used here

causes no contradictions between any observations that might be carried out in different rest frames.

The use of a first-order Lagrangian resolves the obvious question of why photon field theory has a Lagrangian of different order, and a different type of wave-equation, to lepton field theory. The answer is that either type is possible. An advantage is that there is no time-like photon momentum with a negative-metric Hilbert space in this approach, as there is with the Gupta-Bleuler quantization of the covariant Fermi Lagrangian for the quantum Maxwell equations. Thus, the normal axioms of quantum theory apply. Other dual-symmetric covariant Lagrangians are known[28, 29, 36], but are generally more complex, and involve additional auxiliary fields. These auxiliary fields could have a physical interpretation as (for example) relics of quantum gravity. From this perspective, at a more fundamental level, one might then regard the present electromagnetic Lagrangian as an effective quasi-particle Lagrangian for low-energy excitations, analogous to the effective Lagrangians used for phonons.

IV. CONSERVED CURRENTS AND SYMMETRY PROPERTIES

The Lagrangian has a large number of symmetries, resulting in both the usual energy-momentum conservation laws, and additional conserved currents due to internal symmetries. Some of these are far from obvious using the Fermi Lagrangian, and can be more readily derived with complex fields. A particularly interesting internal symmetry is that due to electric-magnetic duality. This is intimately related to photon number, as it generates the conserved helicity number as a local symmetry.

A. Conserved currents

In a free electromagnetic field, the photon number of each mode is conserved, leading to an infinite set of globally conserved quantum numbers. What is surprising is that there is also an infinite set of *locally* conserved densities and currents, as can be readily shown from the complex field equations. These are obtained by first defining an infinite hierarchy of quantum fields $\widehat{\boldsymbol{\mathcal{A}}}^{(n)}$ through the definition $\widehat{\boldsymbol{\mathcal{A}}}^{(1)} = \widehat{\boldsymbol{\mathcal{A}}}$, and the recursion relation:

$$\widehat{\boldsymbol{\mathcal{A}}}^{(n+1)} = \boldsymbol{\nabla} \times \widehat{\boldsymbol{\mathcal{A}}}^{(n)} = i\frac{\partial}{\partial t}\widehat{\boldsymbol{\mathcal{A}}}^{(n)} \ . \tag{4.1}$$

It is easily verified that each field in the hierarchy obeys an identical Maxwell equation, although the higher derivatives of the fields are not usually considered as distinct physical fields. Next, define a normally ordered operator density

$$\widehat{\rho}^{(m,n)} = \frac{1}{2} : \widehat{\boldsymbol{\mathcal{A}}}^{(m)\dagger} \cdot \widehat{\boldsymbol{\mathcal{A}}}^{(n)} : \tag{4.2}$$

and current

$$\widehat{\boldsymbol{J}}^{(m,n)} = \frac{1}{2i} : \widehat{\boldsymbol{\mathcal{A}}}^{(m)\dagger} \times \widehat{\boldsymbol{\mathcal{A}}}^{(n)} : \tag{4.3}$$

for every pair of integers (m, n).

The local conservation of the four-current $\widehat{J}^{(m,n)} = (\widehat{\rho}^{(m,n)}, \widehat{\boldsymbol{J}}^{(m,n)})$ is then proved by using the complex field equation (3.15), which also holds for the quantized fields, so that:

$$\begin{aligned} \frac{\partial}{\partial t}\widehat{\rho}^{(m,n)} + \nabla \cdot \widehat{\boldsymbol{J}}^{(m,n)} &= \frac{i}{2} : \left[\nabla \times \widehat{\boldsymbol{\mathcal{A}}}^{(m)\dagger}\right] \cdot \widehat{\boldsymbol{\mathcal{A}}}^{(n)} : - \\ &\quad -\frac{i}{2} : \left[\nabla \times \widehat{\boldsymbol{\mathcal{A}}}^{(m)\dagger}\right] \cdot \widehat{\boldsymbol{\mathcal{A}}}^{(n)} : + \ldots \\ &= 0 \ . \end{aligned} \tag{4.4}$$

This conservation law holds for every integer pair m, n. Some of these results have also been identified previously[39]. For each integer pair there is a globally conserved operator given by:

$$\widehat{N}^{(m,n)} = \frac{1}{2} \int : \widehat{\boldsymbol{\mathcal{A}}}^{(m)\dagger}(\mathbf{x}) \cdot \widehat{\boldsymbol{\mathcal{A}}}^{(n)}(\mathbf{x}) : d^3\mathbf{x} \tag{4.5}$$

An identical procedure can be used to obtain conservation laws for individual spin densities. Defining a hierarchy of fields in a similar fashion to Eq(4.1)

$$\widehat{\boldsymbol{\mathcal{A}}}_{\sigma}^{(n+1)} = \boldsymbol{\nabla} \times \widehat{\boldsymbol{\mathcal{A}}}_{\sigma}^{(n)} \ , \tag{4.6}$$

where $\sigma = \pm 1$ and $\widehat{\boldsymbol{\mathcal{A}}}_{\sigma}^{(1)} = \widehat{\boldsymbol{\mathcal{A}}}_{\sigma}$, each resulting n-th order spin component field satisfies a Maxwell equation as in Eq (3.6). Just as in the derivation above, this immediately leads to conservation laws for the spin components. The locally conserved spin densities and fluxes are:

$$\begin{aligned} \widehat{\rho}_{\sigma\sigma'}^{(m,n)} &= \frac{1}{2} : \widehat{\boldsymbol{\mathcal{A}}}_{\sigma}^{(m)\dagger} \cdot \widehat{\boldsymbol{\mathcal{A}}}_{\sigma'}^{(n)} : \\ \widehat{\boldsymbol{J}}_{\sigma\sigma'}^{(m,n)} &= \frac{1}{2i} : \widehat{\boldsymbol{\mathcal{A}}}_{\sigma}^{(m)\dagger} \times \widehat{\boldsymbol{\mathcal{A}}}_{\sigma'}^{(n)} : \ . \end{aligned} \tag{4.7}$$

An important question that arises here, is whether or not the $\widehat{J}^{(m,n}$ should be regarded as *local* observables. This first requires hermiticity, which is immediate for symmetric currents

$\widehat{J}^{(n,n)}$. For more general cases, one must construct quadrature currents through defining:

$$\widehat{J}^{(m,n)} = \widehat{J}_x^{(m,n)} + i\widehat{J}_y^{(m,n)} \tag{4.8}$$

where $\widehat{J}_{x,y}^{(m,n)}$ are individually hermitian.

It is useful to distinguish three types of locality:

Local densities and currents are constructed directly from the fields and their derivatives, rather than from their potentials. This requires that $m > 1$ and $n > 1$, so that only fields occur in the definition.

Quasi-local observables vanish whenever the corresponding physical fields ($\boldsymbol{\mathcal{E}}$ here) and their derivatives vanish. This implies that all four-currents with at least one of m, $n > 1$ are local in this sense.

Non-local observables have no physical fields in their definition. However, in view of the relationship between the Aharonov-Bohm[37] effect and the vector potential, even the cases with m, $n < 2$ (which only involve potentials) may sometimes be usefully regarded as observables. The reason for this is that, in quantum mechanics, loop integrals of these potentials can in principle be observed using quantum interference via the Aharonov-Bohm and related effects.

The fields $\widehat{\boldsymbol{\mathcal{A}}}^{(n)}$ defined here have dimension $[\ell]^{-n}$, and the conserved currents $\widehat{J}^{(m,n)}$ therefore have dimension $[\ell]^{-(m+n)}$. Examples of these conserved quantities are given in the following subsections.

B. Potential density $\widehat{\rho}_P$

The density $\widehat{\rho}_P = \widehat{\rho}^{(1,1)}$ is is a conserved density of vector potentials with dimension $[\ell]^{-2}$, having a global conservation law of dimension length or time. It is potentially observable in the sense of interference measurements, but is otherwise nonlocal with regard to the electric and magnetic fields. In terms of real vector potentials, the density and conserved currents are:

$$\begin{aligned} \widehat{\rho}^{(1,1)} &= \frac{1}{2} : \left(\left| \widehat{\boldsymbol{\Lambda}} \right|^2 + \left| \widehat{\boldsymbol{A}} \right|^2 \right) : \\ \widehat{\boldsymbol{J}}^{(1,1)} &= : \widehat{\boldsymbol{\Lambda}} \times \widehat{\boldsymbol{A}} : \end{aligned} \tag{4.9}$$

It is unusual to have a conservation law involving only potentials, and this does not appear to have been identified previously. Like the total energy, it is positive definite. At present, the physical application of this conserved density remains to be found.

However, one can make some immediate remarks using simple dimensional arguments. This conservation law defines a globally conserved 'potential length' $\widehat{L}_P$ which is invariant during propagation of an electromagnetic field:

$$\widehat{L}_P = \frac{1}{2}\int : \left(\left| \widehat{\mathbf{\Lambda}} \right|^2 + \left| \widehat{\boldsymbol{A}} \right|^2 \right) : d^3\mathbf{x} \,. \tag{4.10}$$

For a monochromatic field with frequency ω and photon number $\widehat{n}$, $\widehat{L}_P = \widehat{n}/\omega$. Thus, it has a different scaling property to the conventional coherence length or time, which are both defined as independent of the photon number. For black-body radiation at temperature T, measured in units with $k_B = 1$, the mean potential density is finite, and is given by analogy with the Stefan-Boltzmann law[13] as:

$$\rho_P = \left\langle \widehat{\rho}^{(1,1)} \right\rangle = \frac{T^2}{6} \,. \tag{4.11}$$

It would be interesting to perform direct detection of the conserved potential current, as with the conserved photon and energy currents. This may not be impossible, perhaps via the use of microscopic Aharonov-Bohm interferometers[38] or similar devices.

C. Helicity density $\widehat{n}_h$

The quasi-local density $\widehat{n}_h = \widehat{\rho}_x^{(2,1)}$ with dimension $[\ell]^{-3}$ corresponds to a dimensionless global conserved quantity, having the interpretation of a quantum number. This can be generated from an important internal symmetry property of the present equations. The Lagrangian has a dual symmetry generating by changing the phase of $\Psi \rightarrow e^{i\epsilon}\Psi$, thus rotating $\mathbf{E}$ into $\mathbf{B}$. This symmetry is similar to the chiral symmetry properties known for Weyl (massless) neutrinos. This will be shown to correspond to the photon helicity number. Using Noether's theorem directly on the covariant Lagrangian (3.14), one finds a conserved current given by:

$$\begin{aligned} J_h^\mu &= i \sum_r \frac{\partial \mathcal{L}_c}{\partial \Psi_{r,\mu}} \Psi_r \\ &= \Psi^\dagger \alpha^\mu \Psi \,. \end{aligned} \tag{4.12}$$

In terms of the real fields, the conserved density be re-expressed in a simple form[39]:

$$\begin{aligned} J_h^0 = \widehat{\rho}_x^{(2,1)} &= \frac{1}{4} : \left(\widehat{\boldsymbol{\mathcal{E}}}^\dagger \cdot \widehat{\boldsymbol{\mathcal{A}}} + \widehat{\boldsymbol{\mathcal{E}}} \cdot \widehat{\boldsymbol{\mathcal{A}}}^\dagger \right) : \\ &= \frac{1}{2} : \left(\widehat{\boldsymbol{\Lambda}} \cdot \widehat{\mathbf{E}} + \widehat{\mathbf{A}} \cdot \widehat{\mathbf{B}} \right) : \quad . \end{aligned} \tag{4.13}$$

The corresponding current is obtained either from the covariant expression (4.12), together with the result that for any two vectors, $\mathbf{u}^T S^j \mathbf{v} = -i \left[\mathbf{u} \times \mathbf{v} \right]^j$, or equivalently from the general conserved current definition, (4.3), giving:

$$\begin{aligned} \mathbf{J}_h = \widehat{\boldsymbol{J}}_x^{(2,1)} &= \frac{1}{4i} : \left(\widehat{\boldsymbol{\mathcal{E}}}^\dagger \times \widehat{\boldsymbol{\mathcal{A}}} + \widehat{\boldsymbol{\mathcal{A}}}^\dagger \times \widehat{\boldsymbol{\mathcal{E}}} \right) : \\ &= \frac{1}{2} : (\widehat{\mathbf{E}} \times \widehat{\mathbf{A}} - \widehat{\mathbf{B}} \times \widehat{\boldsymbol{\Lambda}}) : \quad . \end{aligned} \tag{4.14}$$

By Noether's theorem, this symmetry must generate a globally conserved normally ordered operator:

$$\begin{aligned} \widehat{N}_h = \widehat{N}_x^{(2,1)} &= \int : \widehat{\Psi}^\dagger \alpha^0 \widehat{\Psi} : d^3\mathbf{x} \\ &= \frac{1}{2} \int : [\widehat{\boldsymbol{\Lambda}} \cdot \widehat{\mathbf{E}} + \widehat{\mathbf{A}} \cdot \widehat{\mathbf{B}}] : d^3\mathbf{x} \, . \end{aligned} \tag{4.15}$$

This globally conserved charge can be readily identified as the helicity number of the photon field, which is just the difference between the number of right and left circularly polarized photons. This is clear since, from the mode expansion in Eq (3.10), one can show that:

$$\begin{aligned} \widehat{N}_h &= \sum_\sigma \sigma \int \widehat{a}^\dagger_{\mathbf{k}\sigma} \widehat{a}_{\mathbf{k}\sigma} d^3\mathbf{k} \\ &= \int \left[\widehat{n}_{\mathbf{k}+} - \widehat{n}_{\mathbf{k}-} \right] d^3\mathbf{k} \, . \end{aligned} \tag{4.16}$$

Since the helicity number is a difference operator, it is not positive-definite. This means that it is possible to define a hermitian dual phase operator conjugate to $\widehat{N}_h$, which must therefore have a physical realization corresponding to a phase-difference between two modes of opposite helicity. By comparison, a hermitian quantum phase operator conjugate to the usual positive-definite number operator is *not* well-defined except as a singular limit [7, 8]. This is generally in agreement with the experimental situation, which is that phase-measurements involve two modes, whose relative phase is then conjugate to a photon number difference. The dual phase is an example of this situation.

In summary, the helicity number density operator $\widehat{\rho}_x^{(2,1)} = \widehat{n}_h$ defined here has the useful characteristics that it:

- is a global property of the fields
- has a quasi-local conservation law, and
- corresponds to an exact phase-rotation symmetry of the free-field Lagrangian.

Direct detection of this quantity requires a photo-detector that is sensitive to the helicity sign. Although most current photo-detectors do not make this distinction, it is physically possible to distinguish helicity via excitation from an $m = 0$ atomic state to a pair of degenerate levels having $m = \pm 1$. Owing to selection rules, the measured helicity flux along the quantization axis is then proportional to the population difference in the upper levels.

D. Helicity entanglement density $\widehat{m}_h$

The quasi-local density $\widehat{m}_h = \widehat{\rho}_y^{(2,1)}$ with dimension $[\ell]^{-3}$ corresponds to a dimensionless global conserved quantity, which can also be generated from an internal symmetry property, since it is possible to rescale $\boldsymbol{\mathcal{E}} \to e^{\epsilon}\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{A}} \to e^{-\epsilon}\boldsymbol{\mathcal{A}}$ simultaneously, without changing the Lagrangian. This generates another conserved current which can be termed the helicity entanglement current. By Noether's theorem the 4-current due to the rescaling symmetry is:

$$\widehat{J}_y^{(2,1)} = \widehat{\Psi}^{\dagger}\alpha^{\mu}\alpha^4\widehat{\Psi}\,. \tag{4.17}$$

Here the matrix α^4 is defined as:

$$\alpha^4 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}\,. \tag{4.18}$$

The conserved density in this case is given by:

$$\begin{aligned} \widehat{m}_h = \widehat{\rho}_y^{(2,1)} &= \frac{1}{4i} : \left(\widehat{\boldsymbol{\mathcal{E}}}^{\dagger} \cdot \widehat{\boldsymbol{\mathcal{A}}} - \widehat{\boldsymbol{\mathcal{E}}} \cdot \widehat{\boldsymbol{\mathcal{A}}}^{\dagger}\right) : \\ &= \frac{1}{2} : \left(\widehat{\mathbf{A}} \cdot \widehat{\mathbf{E}} - \widehat{\boldsymbol{\Lambda}} \cdot \widehat{\mathbf{B}}\right) : \quad . \end{aligned} \tag{4.19}$$

Just as in the case of the helicity, a corresponding current is obtained either from the covariant expression (4.17), or from the general conserved current definition:

$$\begin{aligned}\widehat{\boldsymbol{J}}_y^{(2,1)} &= \frac{1}{4}:\left(\widehat{\boldsymbol{\mathcal{A}}}^\dagger\times\widehat{\boldsymbol{\mathcal{E}}}+\widehat{\boldsymbol{\mathcal{A}}}\times\widehat{\boldsymbol{\mathcal{E}}}^\dagger\right):\\ &= \frac{1}{2}:\left(\widehat{\boldsymbol{\Lambda}}\times\widehat{\mathbf{E}}+\widehat{\mathbf{A}}\times\widehat{\mathbf{B}}\right):\ .\end{aligned}\tag{4.20}$$

By Noether's theorem, there is a corresponding globally conserved scaling charge:

$$\begin{aligned}\widehat{N}_y^{(2,1)} &= \int:\widehat{\Psi}^\dagger\alpha^0\alpha^4\widehat{\Psi}:d^3\mathbf{x}\\ &= \frac{1}{2}\int:\left(\widehat{\mathbf{A}}\cdot\widehat{\mathbf{E}}-\widehat{\boldsymbol{\Lambda}}\cdot\widehat{\mathbf{B}}\right):d^3\mathbf{x}.\end{aligned}\tag{4.21}$$

This particular conservation law has unusual properties, in that on using a mode-expansion one finds that the global squeezing is always zero in free space using the transverse gauge choice employed here. However, departures from this are possible locally. To understand this, the entanglement density $\widehat{m}_h(\mathbf{x})$ can be re-expressed in terms of helicity components. There are two main contributions, one from photon-density terms, and another from phase-sensitive terms proportional to $\widehat{a}_{\mathbf{k}\sigma}\widehat{a}_{\mathbf{k}'\sigma'}$ and $\widehat{a}^\dagger_{\mathbf{k}\sigma}\widehat{a}^\dagger_{\mathbf{k}'\sigma'}$. These correspond to a quadrature-squeezing[40, 41] measurement, in which opposite helicities are correlated. This is measurable using the technique of homodyne detection with a local oscillator field. To consider this in greater detail, suppose that the relevant fields are nearly monochromatic at frequency $\omega = k$. From the mode-expansion, (3.10) $\widehat{\boldsymbol{\mathcal{E}}}_\sigma \approx \sigma k\widehat{\boldsymbol{\mathcal{A}}}_\sigma$. In this situation the photon-density terms vanish, and one is left only with phase-sensitive terms:

$$\widehat{m}_h(\mathbf{x}) \approx \frac{k}{2i}\left(\widehat{\boldsymbol{\mathcal{A}}}^\dagger_+\cdot\widehat{\boldsymbol{\mathcal{A}}}_- - \widehat{\boldsymbol{\mathcal{A}}}^\dagger_-\cdot\widehat{\boldsymbol{\mathcal{A}}}_+\right)\ .\tag{4.22}$$

Using the mode expansion again, this can be rewritten in terms of the momentum components as:

$$\widehat{m}_h(\mathbf{x}) \approx \int\int\frac{d^3\mathbf{k}d^3\mathbf{k}'}{i(2\pi)^3}\mathbf{e}^*_{\mathbf{k}+}\cdot\mathbf{e}^*_{\mathbf{k}'-}\left[\widehat{a}^\dagger_{\mathbf{k}+}\widehat{a}^\dagger_{\mathbf{k}'-}e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}-h.c.\right]\ .\tag{4.23}$$

However, this expression corresponds precisely to the phase-sensitive part of a quadrature squeezing measurement, carried out using homodyne detection with local oscillators at wavevectors near $\mathbf{k}$ and $\mathbf{k}'$. In other words, the conserved current is proportional to the squeezing in the quadrature correlations of the radiation field. This involves two distinct modes, as in the measurement of EPR correlations[42]. Since pure-state squeezing in distinct modes is a condition for entanglement[43], this implies that a measurement of $\widehat{m}_h(\mathbf{x})$ can be used as an entanglement criterion.

E. Energy density

This is a conserved density of vector potentials with dimension $[\ell]^{-4}$, having a global conservation law with energy or momentum units. Not unexpectedly in view of its dimensionality, this conserved density is the energy density, and the corresponding current is the Poynting vector or momentum current. These quantities are known from Noether's theorem to be generated from the time and space-translation invariance of the Hamiltonian. In terms of real vector potentials, the density and conserved currents are:

$$\begin{aligned} \widehat{\rho}^{(2,2)} &= \frac{1}{2} : \left(\left| \widehat{\mathbf{E}} \right|^2 + \left| \widehat{\mathbf{B}} \right|^2 \right) : \\ \widehat{\mathbf{J}}^{(2,2)} &= : \widehat{\mathbf{E}} \times \widehat{\mathbf{B}} : \, . \end{aligned} \tag{4.24}$$

These quantities are perhaps the most well-known conserved electromagnetic currents, and it is expected that they are obtained here. The globally conserved quantity is just the energy or Hamiltonian:

$$\widehat{H} = \frac{1}{2} \int : \left(\left| \widehat{\mathbf{E}} \right|^2 + \left| \widehat{\mathbf{B}} \right|^2 \right) : d^3\mathbf{x} \, . \tag{4.25}$$

For a monochromatic field with frequency ω and photon number $\widehat{n}$, $\widehat{H} = \omega \widehat{n}$. For blackbody radiation at temperature T, the energy density is given by the Stefan-Boltzmann law[13]:

$$\mathcal{H} = \left\langle \widehat{\rho}^{(2,2)} \right\rangle = \frac{\pi^2 T^4}{15} \, . \tag{4.26}$$

In addition, one might expect to find the usual angular momentum conservation laws. However, these are not simply homogeneous polynomials of fields, and hence cannot be expressed in the elementary form given by the definition of the conserved densities (4.2). Finally, it should be noted that direct detection of energy flux is well-known, and simply requires efficient energy transfer followed by thermometry.

F. Time-averaged photon densities

In practical photo-detection, it is well-known that photo-detectors measure normally ordered correlation functions[44]. This is caused by their finite-time response functions, which average over an optical cycle. Similar time-averaging behavior is obtained in local-oscillator (homodyne) detectors, which time-average with respect to a fixed external carrier frequency.

For time-averaging detectors like this, one must distinguish the positive and negative frequency field operators. This is easily treated with complex helicity fields, since the helicity potentials $\widehat{\boldsymbol{\mathcal{A}}}_\sigma$ contain frequency components of only one sign: - positive frequencies for positive helicity, and vice-versa.

Recalling that $\widehat{\rho}^{(2,1)}$ can be interpreted as a quasi-local helicity difference operator, the total photon number operator $\widehat{N}$ can be expanded as an integral over a photon density:

$$\begin{aligned} \widehat{N} &= \sum_\sigma \int \widehat{n}_\sigma(\mathbf{x}) d^3\mathbf{x} \\ &= \sum_\sigma \int d^3\mathbf{k} \widehat{n}_{\mathbf{k},\sigma} \, . \end{aligned} \tag{4.27}$$

This leads to a conserved photon density operator defined as

$$\widehat{n}(\mathbf{x}) = \sum_\sigma \widehat{n}_\sigma(\mathbf{x}) \, , \tag{4.28}$$

where:

$$\begin{aligned} \widehat{n}_\sigma(\mathbf{x}) &= \frac{\sigma}{2} \left[\widehat{\rho}^{(1,2)}_{\sigma\sigma} + \widehat{\rho}^{(2,1)}_{\sigma\sigma} \right] \\ &= \frac{\sigma}{4} : \left(\widehat{\boldsymbol{\mathcal{E}}}^\dagger_\sigma \cdot \widehat{\boldsymbol{\mathcal{A}}}_\sigma + \widehat{\boldsymbol{\mathcal{E}}}_\sigma \cdot \widehat{\boldsymbol{\mathcal{A}}}^\dagger_\sigma \right) : \, . \end{aligned} \tag{4.29}$$

Here $\widehat{\boldsymbol{\mathcal{E}}}_\sigma \equiv \widehat{\boldsymbol{\mathcal{A}}}^{(2)}_\sigma$. The normally ordered helicity density $\widehat{n}_h(\mathbf{x})$ is *locally* equal to the difference in the individual spin-component densities only after time-averaging, which removes rapidly oscillating cross-terms proportional to $\widehat{a}_{\mathbf{k}\sigma}\widehat{a}_{\mathbf{k}'\sigma'}$ and $\widehat{a}^\dagger_{\mathbf{k}\sigma}\widehat{a}^\dagger_{\mathbf{k}'\sigma'}$:

$$\widehat{n}_h(\mathbf{x}) \approx \widehat{n}_+(\mathbf{x}) - \widehat{n}_-(\mathbf{x}) \, . \tag{4.30}$$

Thus, while dual symmetry generates a global conservation law corresponding to the helicity number, it is also associated - after time-averaging - with a local density operator for the difference in the spin densities. While definitions of photon density are generally not unique[45], this particular definition has the property that it is related to a well-defined conservation law obtained from a fundamental symmetry of the Lagrangian.

The construction of the photon density operator from positive and negative frequency parts of the total operator, means that $\widehat{n}(\mathbf{x})$ corresponds physically to a detector that cycle-averages the input field to obtain the photon-number. A detector of this type implicitly involves time-averaging. Because of this, the *instantaneous* values of $\widehat{n}(\mathbf{x})$ and $\widehat{n}_\sigma(\mathbf{x})$ are not

positive-definite. However, the time-average of $\widehat{n}(\mathbf{x})$ on time-scales longer than any relevant period *is* positive-definite. Physically this means that any attempt to measure the conserved photon flux on time-scales less than a cycle-period would cause dark-counts, requiring a subtraction to give a quantity corresponding to $\widehat{n}(\mathbf{x})$; thus leading to a non-positive-definite result. These effects are usually not observed, due to the very slow response-times of most current photo-detectors (especially efficient ones), relative to an optical cycle. The related question of photon position, in a state containing *exactly* one photon, has also been analyzed recently[46].

This definition of the total photon number density operator is obtained in terms of quasi-local fields. Unlike the Landau-Peierls[9] or Cook[10] photon density, it does not give a finite photon-number density at locations where the physical fields are all zero. It has a spatial integral which corresponds exactly to the total photon number. However, as is usual in any attempt to construct a photon-number density, other desirable properties - in this case, a positive-definite expectation value - are not present. The situation here is similar to that found in other relativistic quantum field theories, since the most fundamental relevant conserved current is the helicity current, whose density can have either sign. In practical terms, a photo-detector usually resides beneath a surface that is arranged to intercept an input beam. Therefore, it responds to a time and area integrated photon flux, which is much more well-defined.

V. INTERACTIONS

A number of possible Lagrangians can be used to describe the coupling of matter to the radiation field. For simplicity, consider the non-relativistic case. This coupling can then be most simply treated using the complex gauge field $\boldsymbol{\mathcal{A}}$ rather than the relativistic Lagrangian. The treatment will initially be a classical one, but can be readily quantized following the Lagrangian quantization procedures already outlined for the free field.

A. Coupling to an external current

The microscopic electromagnetic coupling of a 4-current $J = (\rho, \mathbf{J})$ in Maxwell's equations modifies the equations for $\mathbf{E}$. It is always possible to divide the charge current $\mathbf{J}$ into

transverse and longitudinal parts $\mathbf{J}^{\perp}$ and $\mathbf{J}^{\parallel}$, with a corresponding notation for the fields, so that Maxwell's equations become:

$$\begin{aligned} \nabla \cdot \mathbf{E}^{\parallel} &= \rho \\ \frac{\partial}{\partial t}\mathbf{E}^{\perp} &= \nabla \times \mathbf{B} - \mathbf{J}^{\perp} \ . \end{aligned} \tag{5.1}$$

The resulting field equations for the complex electromagnetic field are therefore:

$$\begin{aligned} \nabla \cdot \boldsymbol{\mathcal{E}}^{\parallel} &= \rho \\ i\frac{\partial}{\partial t}\boldsymbol{\mathcal{E}}^{\perp} &= \nabla \times \boldsymbol{\mathcal{E}} - i\mathbf{J}^{\perp} \, . \end{aligned} \tag{5.2}$$

The corresponding interaction energy with the external charges and currents is:

$$\mathcal{H}_I = \frac{1}{2}\left|\mathbf{E}^{\parallel}\right|^2 - \mathbf{J}\cdot\mathbf{A} \, . \tag{5.3}$$

However, the longitudinal part of the Coulomb energy can be included in the free energy density $\mathcal{H}_o$, just by replacing $\mathbf{E}^{\perp}$ with the total electric field $\mathbf{E}$. The total energy density, including the Coulomb energy, is therefore:

$$\begin{aligned} \mathcal{H}_J &= \mathcal{H}_o\left(\boldsymbol{\mathcal{E}}^{\perp}\right) + \mathcal{H}_I \\ &= \mathcal{H}_o\left(\boldsymbol{\mathcal{E}}\right) - \mathbf{J}\cdot\mathbf{A} \ . \end{aligned} \tag{5.4}$$

A complex field Lagrangian density that satisfies the requirements of generating the current-coupled equation of motion (5.1) together with the correct energy density is:

$$\mathcal{L}_J = \mathcal{L}_0\left(\boldsymbol{\mathcal{A}}, \dot{\boldsymbol{\mathcal{A}}}\right) - \mathcal{H}_I \ . \tag{5.5}$$

Similarly, one can define a Lorenz covariant form of the equation by introducing a potential $\mathbf{K}$ for the transverse current so that $\mathbf{J}^{\perp} = \nabla \times \mathbf{K}$, with the result that:

$$i\partial_{\mu}\alpha^{\mu}\Psi = -iJ_{\Psi} \ , \tag{5.6}$$

where the current J_{Ψ} is defined as a six-component vector

$$J_{\Psi} = \frac{1}{2}\begin{bmatrix} \mathbf{K} \\ \mathbf{J}^{\perp} \end{bmatrix} \ . \tag{5.7}$$

In this case, the covariant interacting Lagrangian with an external current is:

$$\mathcal{L}_{CJ} = \Psi^{\dagger}p_{\mu}\alpha^{\mu}\Psi - \mathcal{H}_I \ . \tag{5.8}$$

In summary, standard minimal coupling results are regained in the case of an external prescribed current density.

B. Minimal coupling with non-relativistic particles

The overall Lagrangian and Hamiltonian in the Coulomb gauge needs to also include terms that describe the response of the current $\mathbf{J}$ to the imposed fields, which were neglected in the above derivation. In the case that the current $\mathbf{J}$ is due to non-relativistic charged particles, the minimal coupling theory will be explained in detail. In this situation the current density is written explicitly (for charges q_p) as

$$\mathbf{J} = \sum_p q_p \dot{\mathbf{x}}_p \delta\left(\mathbf{x} - \mathbf{x}_p\right) \ . \tag{5.9}$$

The total non-relativistic, minimal coupling Lagrangian is then (for masses m_p)

$$L_N = L_J + \frac{1}{2}\sum_p m_p \dot{\mathbf{x}}_p^2 \ . \tag{5.10}$$

Here,

$$L_J = \int d^3\mathbf{x} \mathcal{L}_J(\mathbf{x}) \tag{5.11}$$

is the total current-coupled Lagrangian. The resulting canonical particle momentum for the p-th charged particle has the usual minimal coupling form:

$$\boldsymbol{\pi}_p = m_p \dot{\mathbf{x}}_p + q_p \mathbf{A}(\mathbf{x}_p) \ . \tag{5.12}$$

In order to demonstrate that the Lagrangian is correct, it can be immediately seen that Maxwell's equations are recovered, together with the correct total energy in the form:

$$\begin{aligned} H_N &= \frac{1}{2}\int \left[|\mathbf{E}|^2 + |\mathbf{B}|^2\right] d^3\mathbf{x} + \frac{1}{2}\sum_p m_p \dot{\mathbf{x}}_p^2 \\ &= \frac{1}{2}\int \left[|\mathbf{E}|^2 + |\mathbf{B}|^2\right] d^3\mathbf{x} + \sum_p \frac{1}{2m_p}[\boldsymbol{\pi}_p - q_p \mathbf{A}(\mathbf{x}_p)]^2 \end{aligned} \tag{5.13}$$

It is also necessary that the full Lorenz equation for the particles are obtained, for this version of minimal coupling to be a correct theory. The p-th particle equations are:

$$m\ddot{x}_{j,p} = q\left[E_j^{\parallel}\left(\mathbf{x}_p\right) + \dot{x}_{i,p}\frac{\partial}{\partial x_{j,p}}A_i\left(\mathbf{x}_p\right) - \frac{d}{dt}A_j\left(\mathbf{x}_p\right)\right] \tag{5.14}$$

Here the static Coulomb term $\mathbf{E}^{\parallel}$ arises from the fact that the longitudinal Coulomb energy can be written as:

$$\begin{aligned} \frac{1}{2}\int \left|\mathbf{E}^{\parallel}\left(\mathbf{x}\right)\right|^2 d^3\mathbf{x} &= \sum_p q_p \phi(\mathbf{x}_p) \\ &= \sum_{p \neq p'} \frac{q_p q_{p'}}{8\pi\left|\mathbf{x}_p - \mathbf{x}_{p'}\right|}. \end{aligned} \tag{5.15}$$

The field ϕ is the usual scalar potential, and the self-energy term is ignored. A subtle problem does arise in evaluating the transverse forces. The magnetic part reduces to the usual expression of $\dot{\mathbf{x}} \times \mathbf{B}$ as expected. The remaining term is not the expected expression for the transverse electric field, which is $\mathbf{E}^{\perp} = \nabla \times \mathbf{\Lambda}$ for the dual potential used here.

This, in fact, causes no problems, provided $\boldsymbol{\mathcal{A}}$ is constrained to be transverse, and vanishes on the boundaries. The field equation for $\boldsymbol{\mathcal{E}}$ shows that $\nabla \times \dot{\mathbf{A}} = -\nabla \times \mathbf{E}^{\perp}$. This means that $\dot{\mathbf{A}} = -\mathbf{E}^{\perp}$, since any longitudinal parts of either field are zero; and spatially constant terms must vanish. Thus, the overall result is that the particle motion obeys the usual Lorenz equation in the form:

$$m_p \ddot{\mathbf{x}}_p = q_p \left(\mathbf{E}\left(\mathbf{x}_p\right) + \dot{\mathbf{x}}_p \times \mathbf{B}\left(\mathbf{x}_p\right) \right) \ . \tag{5.16}$$

In summary, the use of a first order complex Lagrangian allows for the minimal coupling of the field to charged particles in a straightforward way, entirely analogous to the standard development with the second order Lagrangian.

C. Minimal coupling with Dirac particles

Similarly, a relativistic Lagrangian for QED has the form:

$$\mathcal{L}_Q = i\Psi^{\dagger}\partial_{\mu}\alpha^{\mu}\Psi + i\psi^{\dagger}\partial_{\mu}\gamma^{\mu}\psi - \mathcal{H}_I \ . \tag{5.17}$$

Here the field ψ is the usual Dirac electron field (and γ^{μ} are the Dirac gamma matrices), while the interaction term $\mathcal{H}_{int}$ is composed of two parts, namely the local field-particle coupling, and the nonlocal Coulomb terms;

$$\mathcal{H}_I = \frac{1}{2}|\mathbf{E}^{\|}|^2 - q\psi^{\dagger}\partial_i\psi \cdot \mathbf{A}^i \ . \tag{5.18}$$

In this form, we can see that interactions can be readily included, and it is straightforward to verify that the usual Maxwell-Dirac equations and Hamiltonian density is recovered.

It is clear that the equations, although relativistic, are no longer explicitly Lorenz-covariant in this formulation of the QED interaction Lagrangian.

VI. MULTI-POLAR GAUGE

In practical uses of QED for bound charges, it is often useful to consider part of the current as free, and part as bound. This idea was first introduced by Goeppert-Mayer[11], and then extended and greatly generalized by the work of Power and Zienau[12]. The main advantage of this approach compared to minimal coupling, is that multi-photon processes are often much simpler to calculate, and causality is more readily obvious. The use of a bound current allows a more rapidly convergent treatment of magnetization and polarization effects than the simple minimal coupling approach in the previous section.

Multi-polar coupling has a very simple interpretation in the present problem. It simply means that all bound charges are to be included in a polarization $\mathbf{P}$ and magnetization $\mathbf{M}$, which are independent dynamical variables derived from particle coordinates. For this reason, in this section a transformation is carried out from the complex minimal coupling field $\boldsymbol{\mathcal{E}}$ to a complex multipolar field $\boldsymbol{\mathcal{D}}$ defined in terms of the displacement field $\mathbf{D}$, which includes the polarization due to the bound charges. To avoid notational complexity, a complex vector potential is introduced that generates the complex multipolar field, so that $\boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{A}}'$. The net bound charge is assumed to be zero. From this perspective, the multipolar approach is obtained through a straightforward variable change. If, in addition, there are no free charges, then the dual notation is very clear.

By considering the equations in this form, the need for any explicit Coulomb interaction term is eliminated. A companion paper will show that this greatly simplifies the treatment of systems involving both macroscopic dielectrics and atomic resonances, which is a common problem in cavity electrodynamics, quantum optics and photonics.

A. Lagrangian for macroscopic Maxwell equations

The issue of constructing a complex field Lagrangian for the macroscopic form of Maxwell's equations is considered here. In this simple case, the current, magnetization and polarization fields are prescribed externally to the interacting fields. The back-reaction of the fields on the charges will be included in the next subsection.

Maxwell's equations in their macroscopic form[47] when there are bound charges present,

together with a free charge density ρ_f and current $\mathbf{J}_f$, are written:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_f \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \\ \nabla \times \mathbf{H} &= \mathbf{J}_f + \dot{\mathbf{D}} \ , \end{aligned} \tag{6.1}$$

with the standard definitions that $\mathbf{D}$ is the displacement field, $\mathbf{P}$ the polarization field, $\mathbf{M}$ the magnetization and $\mathbf{H}$ the magnetic field. These are related by the definitions:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{P} \\ \mathbf{H} &= \mathbf{B} - \mathbf{M} \ . \end{aligned} \tag{6.2}$$

Maxwell's equations can immediately be rewritten in a form that is more suggestive of the complex field equations:

$$\begin{aligned} \dot{\mathbf{D}} &= \nabla \times [\mathbf{B} - \mathbf{M}] - \mathbf{J}_f \\ \dot{\mathbf{B}} &= \nabla \times [\mathbf{P} - \mathbf{D}] \ . \end{aligned} \tag{6.3}$$

Comparing these equations with the microscopic equations, (5.1), the total current can be readily divided into polarization and magnetization components, as long as the particles are spinless:

$$\mathbf{J} = \mathbf{J}_f + \dot{\mathbf{P}} + \nabla \times \mathbf{M} \ . \tag{6.4}$$

It is simplest to obtain a complex field equation by introducing the complex displacement field $\boldsymbol{\mathcal{D}} = \mathbf{D} + i\mathbf{B}$, which satisfies the equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{\mathcal{D}} &= \rho_f \\ i\frac{\partial}{\partial t}\boldsymbol{\mathcal{D}}^{\perp} &= \nabla \times (\boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{P}}) \ , \end{aligned} \tag{6.5}$$

where the complex polarization field $\boldsymbol{\mathcal{P}}$ is defined as:

$$\boldsymbol{\mathcal{P}} = \mathbf{P} + i\mathbf{M} + i\mathbf{K}_f \ . \tag{6.6}$$

This displacement field equation is similar to the complex field equation (5.2) with an imposed current, except that the curl of the complex polarization $\nabla \times \boldsymbol{\mathcal{P}}$ now appears in the place of $i\mathbf{J}^{\perp} = i\nabla \times \mathbf{K}_f$.

It is important to note that the source term in the equation for the longitudinal part of $\mathbf{D}$ only includes the free charges. In many cases of interest, there are no free charges, and the displacement field is then purely transverse. This eliminates the apparently non-causal behaviour included in the instantaneous Coulomb field, and replaces it by terms originating in causal photon exchange, via the complex transverse field $\boldsymbol{\mathcal{D}}$.

It is desirable to derive this macroscopic field equation from a Lagrangian, as in the minimal coupling case. This is achieved in the most straightforward way by simply making a variable change in the current-coupled Lagrangian, Eq(5.5). Define a complex displacement field potential $\boldsymbol{\mathcal{A}}'$ where:

$$\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}} + \boldsymbol{\chi} \ , \tag{6.7}$$

and $\mathbf{P}^{\perp} = \nabla \times \boldsymbol{\chi}$. In terms of the displacement potential, the current-coupled Lagrangian becomes:

$$\mathcal{L}_J = \frac{i}{2}\left(\dot{\boldsymbol{\mathcal{A}}}' - \dot{\boldsymbol{\chi}}\right) \cdot \nabla \times (\boldsymbol{\mathcal{A}}'^{*} - \boldsymbol{\chi}) - \frac{1}{2}\left|\boldsymbol{\mathcal{E}}\right|^2 + \mathbf{J} \cdot \mathbf{A} \tag{6.8}$$

This can be simplified on noting that a Lagrangian density can be integrated by parts, and is invariant under addition of a total time-derivative. Thus, the term $\dot{\boldsymbol{\chi}} \cdot \nabla \times \boldsymbol{\chi}$ can be removed, as it corresponds to a total derivative after partial integration (using the vector identity that $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$)). Similarly, $\dot{\boldsymbol{\mathcal{A}}}' \cdot \nabla \times \boldsymbol{\chi} \rightarrow -\dot{\boldsymbol{\chi}} \cdot \nabla \times \boldsymbol{\mathcal{A}}'$. These changes cancel the coupling to the polarization current, so that the transformed Lagrangian is:

$$\mathcal{L}' = \frac{i}{2}\dot{\boldsymbol{\mathcal{A}}}' \nabla \times \boldsymbol{\mathcal{A}}'^{*} - \frac{1}{2}\left|\boldsymbol{\mathcal{E}}\right|^2 + \mathbf{J}' \cdot \mathbf{A} \ , \tag{6.9}$$

where $\mathbf{J}' = \nabla \times \mathbf{M} + \mathbf{J}_f$ includes only the free and magnetization part of the current.

Next, since $\left|\boldsymbol{\mathcal{E}}\right|^2 = \left|\boldsymbol{\mathcal{D}}^{\perp}\right|^2 + \left|\mathbf{E}^{\parallel}\right|^2 - 2\mathbf{P}^{\perp} \cdot \mathbf{D}^{\perp} + \frac{1}{2}\left|\mathbf{P}^{\perp}\right|^2$, the Lagrangian is rewritten in the form:

$$\mathcal{L}' = \mathcal{L}_0\left(\boldsymbol{\mathcal{A}}', \dot{\boldsymbol{\mathcal{A}}}'\right) - \mathcal{H}'_I \, , \tag{6.10}$$

where:

$$\mathcal{H}'_I = \frac{1}{2}\left[\left|\mathbf{E}^{\parallel}\right|^2 + \left|\mathbf{P}^{\perp}\right|^2\right] - Re\left(\boldsymbol{\mathcal{P}}^{*} \cdot \boldsymbol{\mathcal{D}}\right). \tag{6.11}$$

This directly generates the complex displacement field equations.

The total Hamiltonian density can be rewritten in a form that involves a free Hamiltonian for the complex displacement field together with an interaction energy:

$$\mathcal{H}' = \frac{1}{2}\left|\boldsymbol{\mathcal{D}}^{\perp}\right|^2 + \mathcal{H}'_I. \tag{6.12}$$

This new interaction term is easily changed to the more usually recognized form, on partial integration. If boundary terms are neglected, then:

$$\mathcal{H}'_I = \frac{1}{2}\left[\left|\mathbf{E}^{\parallel}\right|^2 + \left|\mathbf{P}^{\perp}\right|^2\right] - \mathbf{P}\cdot\mathbf{D}^{\perp} - \mathbf{M}\cdot\mathbf{B} - \mathbf{J}_f\cdot\mathbf{A}\ . \tag{6.13}$$

After quantization, $\mathbf{D}^{\perp}$ and $\mathbf{B}$ can be expanded as previously in annihilation and creation operators. Since the only terms that involve field derivatives are in the new free-field Lagrangian $\mathcal{L}_0\left(\boldsymbol{\mathcal{A}}', \dot{\boldsymbol{\mathcal{A}}}'\right)$, their commutators are identical to the previous commutators for $\mathbf{E}^{\perp}$ and $\mathbf{B}$ — except that $\mathbf{D}^{\perp}$ now includes the bound currents as a polarization term.

B. Multipolar transformation with non-relativistic particles

The above results illustrate how one can define canonical coordinates in terms of the displacement field through a simple variable change. In order to show how to obtain this type of formalism from a microscopic approach, it is necessary to turn to the multipolar transformation of Power and Zienau[12]. With the first-order Lagrangian technique used here, the transformation is again obtained from a variable change, starting from the original minimal-coupling Hamiltonian: just as in the previous section.

The polarization and magnetization fields, $\mathbf{P}$ and $\mathbf{M}$, can be written explicitly in terms of (spinless) particle coordinates of the bound charges[48]. For simplicity, consider the case when the polarization and magnetization integrals are all referred to a common center, taken to be the origin. This is not strictly necessary, but reduces the notational complexity, giving:

$$\begin{aligned}\mathbf{P}(\mathbf{x}) &= \sum_p{}' q_p \mathbf{x}_p \int_0^1 d\lambda \delta^{(3)}(\mathbf{x} - \lambda\mathbf{x}_p) \\ \mathbf{M}(\mathbf{x}) &= \sum_p{}' q_p \boldsymbol{\theta}_p(\mathbf{x}) \times \dot{\mathbf{x}}_p,\end{aligned} \tag{6.14}$$

where $\boldsymbol{\theta}_p$ is a radially weighted integral, with:

$$\boldsymbol{\theta}_p(\mathbf{x}) = \mathbf{x}_p \int_0^1 d\lambda\lambda\delta^{(3)}(\mathbf{x} - \lambda\mathbf{x}_p)\,. \tag{6.15}$$

Here, as earlier, $\mathbf{x}_p$ is the p-th bound charge position, while the notation $\sum_p'$ indicates a restricted sum over only the bound charges.

It is useful to start with the minimal coupling Lagrangian and transform this to multipolar form, including interactions with the atoms. Let $\mathbf{E}$ be the original electric field (excluding polarisation terms), and $\mathbf{D} = \mathbf{E} + \mathbf{P}$ the displacement field, where $\mathbf{P}$ now has the closed form expression given above.

The transverse component of $\mathbf{P}$ can always be written as $\mathbf{P}^\perp = \nabla \times \boldsymbol{\chi}$, where $\boldsymbol{\chi}$ is a vector potential for the polarisation. Just as previously, the variable change involves defining a displacement potential $\boldsymbol{\mathcal{A}}'$, which includes the polarisation term $\boldsymbol{\chi}$ so that $\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}} + \boldsymbol{\chi}$.

In terms of $\boldsymbol{\mathcal{A}}'$, the non-relativistic minimal coupling Lagrangian L_N can be written:

$$\begin{aligned} L_N \;=\; & \frac{1}{2}\sum_p m_p \dot{\mathbf{x}}_p^2 + \int \left[\mathbf{J}\cdot\mathbf{A} - \mathcal{H}_0\left(\boldsymbol{\mathcal{E}}\right)\right] dV \\ & + \frac{i}{2}\int \left(\dot{\boldsymbol{\mathcal{A}}}' - \dot{\boldsymbol{\chi}}\right)\cdot \nabla \times \left(\boldsymbol{\mathcal{A}}'^* - \boldsymbol{\chi}\right) dV \;. \end{aligned} \tag{6.16}$$

Just as for the external current case treated previously, this expression can be immediately transformed to the equivalent multipolar form:

$$\begin{aligned} L_N' \;=\; & \frac{1}{2}\sum_p m_p \dot{\mathbf{x}}_p^2 + \int \left[\mathbf{J}'\cdot\mathbf{A} - \mathcal{H}_0\left(\boldsymbol{\mathcal{E}}\right)\right] dV \\ & + \frac{i}{2}\int \dot{\boldsymbol{\mathcal{A}}}'\cdot \nabla \times \boldsymbol{\mathcal{A}}'^* dV \;. \end{aligned} \tag{6.17}$$

Here $\mathbf{J}'$ is defined again as

$$\begin{aligned} \mathbf{J}' \;&=\; \mathbf{J} - \nabla\times\dot{\boldsymbol{\chi}} \\ &=\; \mathbf{J}_f + \mathbf{J}_m \\ &=\; \mathbf{J} - \mathbf{J}_p \;, \end{aligned} \tag{6.18}$$

and terms like $\dot{\boldsymbol{\chi}}\cdot\nabla\times\boldsymbol{\chi}$ which correspond to total derivatives are neglected.

At first, this does not seem identical to the previous expression for the multipolar Lagrangian L_J'. The field energy term can be simplified, because:

$$\int \mathcal{H}_0\left(\boldsymbol{\mathcal{E}}\right) dV = \int \left[\mathcal{H}_0\left(\boldsymbol{\mathcal{D}}^\perp\right) + \frac{1}{2}\left|\mathbf{P}^\perp\right|^2 + \frac{1}{2}\left|\mathbf{E}^\parallel\right|^2 - \mathbf{P}\cdot\mathbf{D}^\perp\right] dV \tag{6.19}$$

The Lagrangian is then:

$$L_N' = L_0' + L_a' + \int Re\left(\boldsymbol{\mathcal{P}}^*\cdot\boldsymbol{\mathcal{D}}\right) dV \;, \tag{6.20}$$

where

$$\boldsymbol{\mathcal{P}} = \mathbf{P} + i\mathbf{M} + i\mathbf{K}_f \ . \tag{6.21}$$

The first term is simply the free-field Lagrangian written in terms of the displacement rather than the electric field:

$$L_0' = \frac{1}{2}\int \left[i\dot{\boldsymbol{\mathcal{A}}}' \cdot \nabla \times \boldsymbol{\mathcal{A}}'^{*} - \left|\nabla \times \boldsymbol{\mathcal{A}}'\right|^2 \right] dV \ . \tag{6.22}$$

This therefore can be quantized to give annihilation and creation operators following the procedure outlined previously for a free electromagnetic field. These operators should now be regarded as multipolar operators, since they diagonalize a Hamiltonian constructed from the multipolar fields. There is also a new term present which describe the Lagrangian of the bound charges, so that L_a' now describes the kinetic energy and the total energy of the bound charges, including interaction energies with any longitudinal fields present and a transverse polarization self-energy term:

$$L_a' = \frac{1}{2}\sum_p m_p \dot{\mathbf{x}}_p^2 - \frac{1}{2}\int \left[\left|\mathbf{E}^{\parallel}\right|^2 + \left|\mathbf{P}^{\perp}\right|^2 \right] dV \tag{6.23}$$

The field energy terms, as in the conventional Power-Zienau approach, includes two distinct contributions. The first is the longitudinal field energy, which gives the internal Coulomb energy of the dipoles that make up the polarisation field. The second is the transverse polarization energy, which is a component of the Lamb shift. A large part of this term (which is ultra-violet divergent), is cancelled by a similar divergent contribution from the renormalisation of the field-atom coupling.

Finally, the last term describes the multipolar interaction of the fields and charged particles present, including all orders of multipolar electric and magnetic interaction terms. This is conventionally obtained as a multipolar series through an expansion of the previous integral expressions for magnetization and polarization in Eq. (6.14). Thus, for example, in the commonly used electric-dipole approximation, one obtains (after quantization):

$$\widehat{\mathbf{P}}\left(\mathbf{x}\right) \approx \sum_p{}' q_p \widehat{\mathbf{x}}_p \delta^{(3)}\left(\mathbf{x}\right) \ , \tag{6.24}$$

and the corresponding interaction Hamiltonian density is:

$$\widehat{\mathcal{H}}_I = -\widehat{\mathbf{P}} \cdot \widehat{\mathbf{D}}^{\perp} \ . \tag{6.25}$$

In this approximation, the interaction term in the Lagrangian has no field or coordinate derivatives, so it has no effect on the quantization procedure, which is carried out simply using the two noninteracting Lagrangians for the charged particles and transverse fields respectively. The essential physics of the final results given above is identical to that in the earlier Power-Zienau approach. However, this appears to be a more transparent way to obtain the dipole-coupled results, since the change from electric field to displacement field is achieved through a simple variable change in the Lagrangian. In addition, it is rather clear how intermediate forms of the interaction can be obtained, since the results depend entirely on the choice of terms to be included in the bound current. By choosing to include more or less of the total current within $\mathbf{J}_p$, a number of equivalent Lagrangian forms can be generated. These have useful applications to the problem of counter-rotating terms in photon-atom interactions[14].

C. Nonlinear optics in the multipolar gauge

An important topic is the issue of quantization of atom-field interactions in the presence of linear and nonlinear dielectric and magnetic media. Although this will be treated elsewhere in detail, the starting point will be summarized here. In many cases it is useful to consider the electromagnetic medium as having instantaneous response functions $\mathbf{E}[\mathbf{D}]$ and $\mathbf{H}[\mathbf{B}]$, together with external current sources. This is typically the case when all atomic or molecular transitions are far from resonance with the frequencies of interest, so that the relevant atomic response can be adiabatically eliminated.

This, incidentally, is different to the situation where the polarization and magnetization was assumed to be externally prescribed, and only the field energy was included in the Hamiltonian. The concept that the polarization field - and hence the electric field - is a functional of the displacement, is a natural consequence of the macroscopic form of Maxwell's equations, in which the fundamental canonical variables involve the displacement field. This idea has been treated previously[49, 50], and forms a natural route to the development of an effective theory of quantized dielectrics and quantized nonlinear optical media.

In this case, it is very important to recognize that adding new terms to the Hamiltonian, without changing the electric field commutators, will not result in the correct Maxwell's equations. This is caused by the fact that Maxwell's equations assume a nondispersive,

instantaneous response in the form considered here. While this cannot be accurate at sufficiently high frequencies - where one must regain the usual microscopic commutators - it is nevertheless a useful approximation in many situations.

It is particularly straightforward to implement this idea with the present approach. The simplest theory neglects dispersion, and therefore should be considered a low-frequency approximation to a full theory of the dielectric. In this case, the nonlinear Hamiltonian is given by the energy as defined through the work done by the external currents and charges to create the electromagnetic and material fields. This is calculated from the nonlinear response functionals, so that:

$$\mathcal{H}_N(\boldsymbol{\mathcal{D}}) = \int_0^{\mathbf{D}} \mathbf{E}[\mathbf{D}'] \cdot d\mathbf{D}' + \int_0^{\mathbf{B}} \mathbf{H}[\mathbf{B}']\, d\mathbf{B}'. \tag{6.26}$$

To include external charges and currents defined by the presence of a prescribed 4-current J_f^μ, the Lagrangian is modified to:

$$\mathcal{L} = \frac{i}{2}\dot{\mathcal{A}}' \cdot \nabla \times \mathcal{A}'^* - \mathcal{H}_N(\boldsymbol{\mathcal{D}}) + \mathbf{J}_f \cdot \mathbf{A} \tag{6.27}$$

where:

$$\begin{aligned} \mathbf{D} &= \mathbf{D}^{\parallel} + \nabla \times \mathbf{\Lambda}' \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{6.28}$$

Here $\mathbf{D}^{\parallel}$ is the solution to $\nabla \cdot \mathbf{D}^{\parallel} = \rho_f$. It is straightforward to check that charge conservation together with this Lagrangian generates the full nonlinear Maxwell equations, in the form

$$\begin{aligned} \dot{\mathbf{D}} &= \nabla \times \mathbf{H}[\mathbf{B}] - \mathbf{J}_f \\ \dot{\mathbf{B}} &= \nabla \times \mathbf{E}[\mathbf{D}] \ . \end{aligned} \tag{6.29}$$

The correct energy density is also obtained, which is

$$\mathcal{H} = \mathcal{H}_N(\boldsymbol{\mathcal{D}}) - \mathbf{J}_f \cdot \mathbf{A}\,. \tag{6.30}$$

The great advantage of the present formalism is the ease with which a consistent Lagrangian is obtained that can generate both the correct nonlinear Maxwell's equations and the correct energy, while still permitting a coupling to external current sources or atomic dipoles.

VII. SUMMARY

The use of a complex field combining electric and magnetic components has a number of interesting features in developing a canonical theory of the radiation field. The Lagrangian has only first order derivatives, and results in first order field equations. A covariant form is possible, resulting in a six-component wave equation, analogous to the Dirac equation. The fields themselves are massless vectors transforming under a nonlocal infinite-dimensional representation of the Lorenz group. This originates in the problem of photon localization. It does not contradict either causality or unitarity.

The complex form makes it transparent to implement dual phase symmetry transformations, which generate the conservation of helicity number. This dual phase can be regarded as the prototype of a physical phase measurement, since it comes from the electric and magnetic fields themselves. We note that the dual phase is best regarded as a relative phase of two modes, which is how it is operationally measured. It is significant that the corresponding conservation law is for a non-positive-definite operator, the helicity. In fact, this is known to be an essential requirement for obtaining a hermitian phase operator. By contrast, the definition of a phase operator for a single mode of the radiation field is known to result in difficulties.

The idea of a photonic wave-function can be explored further with this approach, and has a simple relationship to the total photon number. As is expected for massless particles with spin, it is not possible to define a positive definite photon density operator, since there is no well-defined photon position in all three dimensions simultaneously. However, it is possible to construct a type of particle density which is positive-definite after time-averaging, and has several desirable properties, when compared to previous proposals. An infinite set of local or quasi-local, normally-ordered conserved densities can be generated, including a photon helicity number density and a squeezing density.

In summary, the complex electromagnetic field $\widehat{\Psi}$ has many of the required attributes of a photon field. This should not be taken to imply that there is a well-defined instantaneous photon position operator, which is ruled out from well-known considerations of Lorenz-invariance. In the present case, the construction of the density operator from positive and negative frequency parts of the total operator, means that $\widehat{n}(x)$ corresponds physically to a detector that cycle-averages the input field to obtain the photon-number. A detector of

this type implicitly involves time-averaging. Thus, the *instantaneous* value of $\widehat{n}(\mathbf{x})$ is not guaranteed positive-definite. In physical terms, any attempt to measure the photon number on time-scales less than a cycle-period, introduces vacuum fluctuation effects.

While the fundamental dual symmetry is broken in the presence of charges, the dual-symmetric Lagrangian can be extended to include either free or bound charges, and the transformation to a multi-polar form of the Lagrangian is directly possible. Thus, the presence of charges does not prevent a Lagrangian for the complex electromagnetic field from being readily constructed. The multipolar form is especially straightforward because it involves the displacement field which is divergence-less in the absence of free charges, like the magnetic field. The detailed application of this theory to dielectric problems will be given elsewhere.

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