Abstract

For calculating the ground state of an ultra-cold Fermi gas, the collisional interaction potential $V(r,r')$ can be described using a contact potential approximation (e.g., [3]) while also invoking a momentum, cutoff to correct the ultra-violet divergence. The correctly renormalized theory is cutoff independent [3] and is equivalent to other standard renormalization procedures (e.g., [2]) in limit that the energy cutoff is infinite.

In the dynamic system where a Bragg grating is applied, the Bragg scattering [1] is sensitive (albeit weakly) to the chosen energy cutoff. In the limit that the cutoff becomes infinite, no Bragg scattering occurs in the renormalized theory. Renormalization is, however, only a tool. The non-renormalizability of this problem arises because Bragg scattering measures the momentum behaviour of the pair function, which is ill defined in a renormalized theory. By using a more accurate description of the collisional interaction potential—a separable potential approximation—we are able to give a relatively simple description of the expected behaviour of the Bragg scattering.

References


Bragg Scattering of Cooper Pairs: Beyond the Pseudopotential Approximation

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1 Two-Component Fermi Gas: Hamiltonian

System Hamiltonian

\[ H = \sum_{\alpha} \int \psi_{\alpha}^\dagger(r, t) \mathcal{H}_0 \psi_{\alpha}(r, t) d^3r + H_{\text{scatt}} \]

Single Particle Hamiltonian (no Trap):

\[ \mathcal{H}_0 = -\frac{\hbar^2\nabla^2}{2M} + V_{\text{opt}}(r, t) \]

Spin-preserving optical Bragg grating:

\[ V_{\text{opt}}(r, t) = \frac{1}{2} A \cos(q \cdot r - \omega t), \quad q \text{ parallel to } x\text{-axis} \]

Scattering Hamiltonian—most general translationally invariant form

\[ H_{\text{scatt}} = \frac{1}{2} \sum_{\alpha} \int V(r, r') \psi_{\alpha}^\dagger(R + \frac{r'}{2}, t) \psi_{-\alpha}^\dagger(R - \frac{r'}{2}, t) \psi_{-\alpha}(R - \frac{r}{2}, t) \psi_{\alpha}(R + \frac{r}{2}, t) d^3r d^3r' d^3R, \]
2 Effective Potentials

Only low energy features are important.

Contact Potentials

- **Fermi:** $V(r, r') = V\delta(r - r')$
  Requires a momentum cutoff and renormalization.

- **Huang-Yang, Bruun-Castin-Dum-Burnett:** $V(r, r') = V\delta(r - r')\partial_R R, \quad (R \equiv |r - r'|)$
  Equivalent to a renormalized Fermi contact potential.
• **We find Bragg scattering is not renormalizable!**
  - Two kinds of Bragg scattering
    1. Single particle Bragg scattering
    2. Cooper pair scattering—*new effect*

• Amount of Cooper pair scattering sensitive to higher energy behaviour of the potential.
  ⇒ Depends (weakly) on the renormalization cutoff.

  **As cutoff → ∞**

  *Cooper pair Bragg scattering* → 0

  ⇒ Require a more accurate description than a contact potential.
Separable potential (Yamaguchi, Köhler-Szymańska)

Approximate collisional interaction potential:
\[ V(r, r') = gF(r)F^*(r') , \]

- \( F(r) \) has a finite range \( \sim \sigma \) with \( \int F(r)d^3r = 1 \).
- **Fourier transform**: \( f(k) = \int F(r)e^{-ik\cdot r}d^3r \)
- **Momentum space potential**: \( V(k, k') = gf(k)f^*(-k') \)
- **Strength parameter**: \( g = \frac{T_{2B}(0)}{1 - \alpha T_{2B}(0)} \), where \( \alpha = \frac{M}{(2\pi)^3\hbar^2} \int \frac{|f(k)|^2 d^3k}{k^2} \)
- **Validity**: Normally within a limited energy range around \( E \approx 0 \)
Parametrization of the Separable Potential

- **Gaussian (Köhler-Szymańska):**
  \[ f(k) = \exp \left( -k^2 \sigma_G^2 / 2 \right) \quad \text{with} \quad \sigma_G = 57.383 a_{\text{Bohr}} \]

- **Step Function (our choice):**
  \[ f(k) = \Theta(k_c - |k|) \quad \text{with} \quad k_c = \sqrt{\pi / 2 \sigma_G} = 31.1 k_F \]

**Behaviour of solutions for large \( k_c \)**

- Strength parameter \( g \) now given by
  \[ g = \frac{T_{2B}(0)}{1 - \alpha T_{2B}(0)}, \quad \text{where} \quad \alpha = \frac{k_c M}{(2\pi^2)\hbar^2} \]

- \( T_{2B}(0) \) fixed and *negative* \( \Rightarrow g \to 0 \) as \( k_c \to \infty \)
3 Mean-Field Approximation—Bogoliubov de Gennes Equations

Dominant collisional interaction described using mean-field pairings

\[
\bar{\tilde{W}}_{\alpha}(r, r', t) = \langle \hat{\psi}_{\alpha}^{\dagger}(r, t) \hat{\psi}_{\alpha}(r', t) \rangle, \quad \bar{\Delta}(r, r', t) = \langle \hat{\psi}_{\alpha}(r, t) \hat{\psi}_{-\alpha}(r', t) \rangle.
\]

Heisenberg equations of motion take the form

\[
\frac{i\hbar}{\partial t} \hat{\psi}_{\uparrow, \downarrow}(r, t) = [\mathcal{H}_0 - \mu] \hat{\psi}_{\uparrow, \downarrow}(r, t) + \int W_{\uparrow, \downarrow}(r, r', t) \hat{\psi}_{\uparrow, \downarrow}(r', t) d^3r' \pm \int \Delta(r, r', t) \hat{\psi}_{\uparrow, \downarrow}^{\dagger}(r', t) d^3r',
\]

where

\[
W_{\alpha}(r, r', t) = \int V \left( r' - r + \frac{s}{2}, r - r' + \frac{s}{2} \right) \bar{\tilde{W}}_{\alpha} \left( r' - \frac{s}{2}, r - \frac{s}{2}, t \right) d^3s,
\]

and

\[
\Delta(r, r', t) = \int V(s, r - r') \bar{\Delta}_i \left( \frac{r' + r - s}{2}, \frac{r' + r + s}{2}, t \right) d^3s.
\]
**Quasi-homogeneous approximation**

- BCS correlation length $\gg$ range of the collisional potential.
- Equal population densities of two states,
- Hartree field $\bar{U}(r) \equiv g \langle \psi_1 \uparrow \psi_1 \downarrow \rangle$
- Pair function $\Delta(r) \equiv g \langle \psi \downarrow (r) \psi \uparrow (r) \rangle$
- Using separable form, mean-fields become

$$W_\alpha(r, r', t) \approx \bar{U}(r, t) \int F \left( r' - r + \frac{s}{2} \right) F^* \left( r - r' + \frac{s}{2} \right) d^3 s$$

$$\Delta_\alpha(r, r', t) \approx \Delta(r, t) F^* (r - r')$$
Bogoliubov Quasiparticle Representation

\[ \psi_\uparrow(r, t) = \sum_k \left[ u_k(r, t) \gamma_k \uparrow - v_k^*(r, t) \gamma_k^\dagger \right] \]

\[ \psi_\downarrow(r, t) = \sum_k \left[ u_k(r, t) \gamma_k \downarrow + v_k^*(r, t) \gamma_k^\dagger \right] \]

*Time-independent* quasi-particle Fermion annihilation operators \( \gamma_{k\alpha} \)

\[ \langle \gamma_{k\alpha}^\dagger \gamma_{k'\beta} \rangle = \delta_{kk'} \delta_{\alpha\beta} \bar{n}_k \]

\[ \langle \gamma_{k\alpha} \gamma_{k'\beta} \rangle = 0 \]

Fermi-function:

\[ \bar{n}_k = \frac{1}{\exp(\epsilon_k/k_B T) + 1} \]
Time-dependent Bogoliubov de Gennes equations

- General form:
  \[
  i\hbar \frac{\partial u_k(r, t)}{\partial t} = [\mathcal{H}_0 - \mu] u_k(r, t) + \int W(r, r', t) u_k(r', t) d^3r' + \int \Delta(r, r', t) v_k(r', t) d^3r'
  \]
  \[
  i\hbar \frac{\partial v_k(r, t)}{\partial t} = -[\mathcal{H}_0 - \mu] v_k(r, t) - \int W(r, r', t) v_k(r', t) d^3r' + \int \Delta^*(r, r', t) u_k(r', t) d^3r'
  \]

- Use appropriate separable potential forms for $W$, $\Delta$ collisional interaction potential

- Reproduce the usual ground state BCS equations.
Relation to Standard BCS Theory

Spatially homogeneous:

\[
U(r, t) \rightarrow U, \quad \Delta(r, t) \rightarrow \Delta, \\
u_k(r) \rightarrow a_k^0 e^{i k \cdot r}, \quad v_k(r) \rightarrow b_k^0 e^{i k \cdot r}, \quad |a_k^0|^2 + |b_k^0|^2 = \frac{1}{L^3}
\]

Time-independent BCS equations:

\[
\begin{align*}
\epsilon_k a_k^0 &= \left( E_k - \mu + U \left| f(\frac{k}{2}) \right|^2 \right) a_k^0 + \Delta \left| f^*(k) \right| b_k^0 \\
\epsilon_k b_k^0 &= - \left( E_k - \mu + U \left| f(\frac{k}{2}) \right|^2 \right) b_k^0 + \Delta^* f(k) a_k^0
\end{align*}
\]
Solutions

Standard solution: \( f(k) = \Theta(k_c - |k|) \)

\[
E_k = \frac{\hbar^2 k^2}{2M}
\]

\[
U = g \sum_k \left( |a_k|^2 \bar{n}_k + |b_k|^2 (1 - \bar{n}_k) \right)
\]

\[
\Delta = -g \sum_k a_k^* b_k (1 - 2 \bar{n}_k)
\]

\[
\epsilon_k = \sqrt{(E_k - \mu + U)^2 + |\Delta|^2}
\]

\[
a_k^0 = \eta_k \Delta
\]

\[
b_k^0 = \eta_k (\epsilon_k - E_k + \mu - U)
\]

Renormalizable Solutions

- For a given cutoff \( k_c \), choose the strength \( g \) to give the correct scattering length.

- Same long wavelength physics independent of the cutoff.

- But as \( k_c \to \infty \), \( g \to 0 \), and thus \( U, \Delta \to 0 \)!
  
  BCS physics then expressed in terms of renormalized quantities

  \[
  \tilde{\Delta} = T_{2B}(0) \Delta / g, \quad \tilde{U} = T_{2B}(0) U / g
  \]

- Inclusion of optical Bragg potential \( V_{\text{opt}}(r) \) \( \to \) cutoff dependence
IS IT LEGAL TO HAVE A CUTOFF DEPENDENCE?
What Does Renormalization Do?

- Reduces $a_k$, $b_k$, while increasing the cutoff so that overall sums are preserved.
- Suppresses possibly interesting finite $k$ behaviour in favour of indeterminate high $k$ behaviour
Cutoff Dependence in the BCS Theory of Superconductivity

- Interaction between electrons mediated by crystal lattice
- Natural cutoff $k_{\text{Debye}}$ given by Debye frequency
- Debye frequency depends on masses of crystal atoms
- Leads to the **isotope effect**—critical temperature depends on the isotope of the superconducting metal.
- Experimentally well-verified effect
- Historically was evidence that the crystal lattice was involved in superconductivity
- Renormalization would **destroy** the isotope effect

![Graph](image)

*Fig. 10.15. Isotope effect for tin (Sn). The results of several authors are summarized [10.10]; Maxwell (◇); Lock, Pippard, Shoenberg (■); Serin, Reynolds and Lohman (△).*
4 Bragg Scattering Equations

**Bloch form**

\[ u_k(r, t) = e^{ikr} \sum_n a_n^k(t) e^{in(q \cdot r - \omega t)} , \]
\[ v_k(r, t) = e^{ikr} \sum_n b_n^k(t) e^{in(q \cdot r - \omega t)} , \]
\[ U(r, t) = \sum_n U_n(t) e^{in(q \cdot r - \omega t)} , \]
\[ \Delta(r, t) = \sum_n \Delta_n(t) e^{in(q \cdot r - \omega t)} \]

Initial condition: amplitudes \( u_k(r, t) \) and \( v_k(r, t) \) are the BCS stationary states.
Evolution equations

\[
\begin{align*}
\hbar \frac{\partial a_n^k(t)}{\partial t} &= \hbar \omega_n^a(k) a_n^k(t) + \frac{1}{4} A \left[ a_{n-1}^k(t) + a_{n+1}^k(t) \right] \\
&+ \sum_p \left\{ U_{n-p}(t) |f \left( \frac{1}{2}(k + pq) \right) |^2 a_p^k(t) + \Delta_{n-p}(t) f^*(k + pq) b_p^k(t) \right\} \\
\hbar \frac{\partial b_n^k(t)}{\partial t} &= \hbar \omega_n^b(k) b_n^k(t) - \frac{1}{4} A \left[ b_{n-1}^k(t) + b_{n+1}^k(t) \right] \\
&- \sum_p \left\{ U_{n-p}(t) |f \left( \frac{1}{2}(k + pq) \right) |^2 b_p^k(t) + \Delta^*_{p-n}(t) f(k + pq) a_p^k(t) \right\}
\end{align*}
\]

where

\[
\hbar \omega_n^b(k) = -E_{k+q} + \mu - n\hbar \omega, \quad \hbar \omega_n^a(k) = E_{k+q} - \mu - n\hbar \omega
\]
5 Numerical Results

- Two-component gas
- Material $^{40}$K in the $(F = 9/2, m_F = -9/2)$ and $(F = 9/2, m_F = -7/2)$ Zeeman states
- Cutoff $k_c \sim 31.1k_F$.
- Scattering from initial pair correlations easily distinguished from the scattering in a non-interacting gas.

High momentum transfer

- $|q| = 4.33k_F \Rightarrow q/2$ well outside Fermi surface.
- 4 Bragg orders during the evolution
  - Only $n = 0, \pm 1, -2$, for $a_n^k$ and $b_n^k$
Initial State—BCS Paired Fermi Gas
Bragg Scattering from Fermi Gas in BCS Regime
Interpretation—Correlated-pair Bragg scattering

- Possible in a Cooper paired Fermi gas because of the initial pair correlations.
- Characterized by
  - Spherical shell of correlated atoms in momentum space centred at momentum $\frac{\hbar q}{2}$
  - On the red-detuned side of the single-particle Bragg resonance
  - Slight asymmetry in the Bragg spectra.
- Spherical shell not a direct result of single-particle Bragg scattering events
- Two stage process
  1. Generation of the pair potential grating via single-particle Bragg scattering
  2. Scattering of Cooper pairs via the Bragg grating in the pair potential.
**Quantitative Model**

- Momentum and energy conservation
- Zero COM Cooper pair—initial momenta $\pm \hbar \mathbf{k}_P$.
- Mainly reside on the Fermi surface $\implies |\mathbf{k}_P| \sim k_F'$.
- Cooper pair scattered by the pair potential grating to total COM momentum $\hbar \mathbf{q}$.
- Excess energy distributed equally between the atoms of the pair—final momenta $\hbar (\mathbf{q}/2 \pm \mathbf{k}_{rel})$.
- Energy required $\hbar \omega_{\text{pair}} = \frac{\hbar^2}{M} \left( \frac{q^2}{4} + k_{rel}^2 - k_F'^2 \right)$
- Frequency threshold, $\omega_{\text{thres}} = \frac{\hbar}{M} \left( \frac{q^2}{4} - k_F'^2 \right)$.
- Below threshold frequency the pair potential grating does not have sufficient energy to scatter Cooper pairs.
- Above the threshold scattered pairs form a spherical shell
  
  Momentum radius $k_{rel} \approx \sqrt{\frac{M}{\hbar} (\omega - \omega_{\text{thres}})}$—Agrees well with detailed numerics.