Abstract

For calculating the ground state of an ultra-cold Fermi gas, the collisional interaction potential $V(\mathbf{r}, \mathbf{r}')$ can be described using a contact potential approximation (e.g., [3]) while also invoking a momentum, cutoff to correct the ultra-violet divergence. The correctly renormalized theory is cutoff independent [3] and is equivalent to other standard renormalization procedures (e.g., [2]) in limit that the energy cutoff is infinite.

In the dynamic system where a Bragg grating is applied, the Bragg scattering [1] is sensitive (albeit weakly) to the chosen energy cutoff. In the limit that the cutoff becomes infinite, no Bragg scattering occurs in the renormalized theory. Renormalization is, however, only a tool. The non-renormalizability of this problem arises because Bragg scattering measures the momentum behaviour of the pair function, which is ill defined in a renormalized theory.

By using a more accurate description of the collisional interaction potential—a separable potential approximation—we are able to give a relatively simple description of the expected behaviour of the Bragg scattering.

References

[1] K. J. Challis, R. J. Ballagh, and C. W. Gardiner. Bragg scattering of cooper pairs in an ultracold fermi gas. *Physical Review Letters*, 98(9):093002, 2007.

[2] M. Houbiers, R. Ferwerda, H. T. C. Stoof, W. I. McAlexander, C. A. Sackett, and R. G. Hulet. *Phys. Rev. A*, 56:4864, 1997.

[3] M. L. Chiofalo, S. J. J. M. F. Kokkelmans, J. N Milstein, and M. J. Holland. *Phys. Rev. Lett.*, 88:090402, 2002.

Bragg Scattering of Cooper Pairs: Beyond the Pseudopotential Approximation

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1 Two-Component Fermi Gas: Hamiltonian

System Hamiltonian

$$H = \sum_\alpha \int \psi^\dagger_\alpha(\mathbf{r},t)\mathcal{H}_0\psi_\alpha(\mathbf{r},t)d^3r + H_{\rm scatt}$$

Single Particle Hamiltonian (no Trap):

$$\mathcal{H}_0 = -\frac{\hbar^2\nabla^2}{2M} + V_{\rm opt}(\mathbf{r},t)$$

Spin-preserving optical Bragg grating:

$$V_{\rm opt}(\mathbf{r},t) = \frac{1}{2}A\cos(\mathbf{q}\cdot\mathbf{r}-\omega t), \qquad \mathbf{q} \text{ parallel to } x\text{-axis}$$

Scattering Hamiltonian—most general translationally invariant form

$$H_{\rm scatt} = \frac{1}{2}\sum_\alpha \int V(\mathbf{r},\mathbf{r}')\psi^\dagger_\alpha\left(\mathbf{R}+\frac{\mathbf{r}'}{2},t\right)\psi^\dagger_{-\alpha}\left(\mathbf{R}-\frac{\mathbf{r}'}{2},t\right)\psi_{-\alpha}\left(\mathbf{R}-\frac{\mathbf{r}}{2},t\right)\psi_\alpha\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)d^3r\,d^3r'\,d^3R,$$

2 Effective Potentials

Only low energy features are important.

Contact Potentials

- ***Fermi:*** $V(\mathbf{r}, \mathbf{r}') = V\delta(\mathbf{r} - \mathbf{r}')$

  Requires a momentum cutoff and renormalization.

- ***Huang-Yang, Bruun-Castin-Dum-Burnett:*** $V(\mathbf{r}, \mathbf{r}') = V\delta(\mathbf{r} - \mathbf{r}')\partial_R R, \qquad (R \equiv |\mathbf{r} - \mathbf{r}'|)$

  Equivalent to a renormalized Fermi contact potential.

- ***We find Bragg scattering is not renormalizable!***
  - Two kinds of Bragg scattering
    1. Single particle Bragg scattering
    2. Cooper pair scattering—***new effect***
- Amount of Cooper pair scattering sensitive to higher energy behaviour of the potential.

  $\Longrightarrow$ Depends (weakly) on the the renormalization cutoff.

  As cutoff $\rightarrow \infty$

  Cooper pair Bragg scattering $\rightarrow 0$

  $\Longrightarrow$ Require a more accurate description than a contact potential.

Separable potential (Yamaguchi, Köhler-Szymańska)

Approximate collisional interaction potential:

$$V(\mathbf{r}, \mathbf{r}') = gF(\mathbf{r})F^*(\mathbf{r}'),$$

- $F(\mathbf{r})$ has a finite range $\sim \sigma$ with $\int F(\mathbf{r}) d^3r = 1$.
- ***Fourier transform:*** $f(\mathbf{k}) = \int F(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r$
- ***Momentum space potential:*** $\mathcal{V}(\mathbf{k}, \mathbf{k}') = gf(\mathbf{k})f^*(-\mathbf{k}')$
- ***Strength parameter:*** $g = \dfrac{T_{2B}(0)}{1 - \alpha T_{2B}(0)},$ where $\alpha = \dfrac{M}{(2\pi)^3 \hbar^2} \displaystyle\int \frac{|f(\mathbf{k})|^2 d^3k}{k^2}$
- ***Validity:*** Normally within a limited energy range around $E \approx 0$

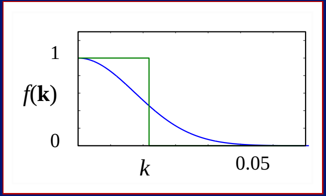
Parametrization of the Separable Potential

- ***Gaussian (Köhler-Szymańska):***

  $f(\mathbf{k}) = \exp\left(-k^2\sigma_G^2/2\right)$ with $\sigma_G = 57.383 a_{\mathrm{Bohr}}$

- ***Step Function (our choice):***

  $f(\mathbf{k}) = \Theta\left(k_c - |\mathbf{k}|\right)$ with $k_c = \sqrt{\pi}/2\sigma_G = 31.1 k_F$



Behaviour of solutions for large k_c

- Strength parameter g now given by

$$g = \frac{T_{2B}(0)}{1 - \alpha T_{2B}(0)}, \quad \text{where} \quad \alpha = \frac{k_c M}{(2\pi^2)\hbar^2}$$

- $T_{2B}(0)$ fixed and ***negative*** $\Longrightarrow g \to 0$ as $k_c \to \infty$

3 Mean-Field Approximation—Bogoliubov de Gennes Equations

Dominant collisional interaction described using mean-field pairings

$$\boxed{\bar{W}_\alpha(\mathbf{r},\mathbf{r}',t)} = \langle \hat{\psi}^\dagger_\alpha(\mathbf{r},t)\hat{\psi}_\alpha(\mathbf{r}',t)\rangle, \qquad \boxed{\bar{\Delta}(\mathbf{r},\mathbf{r}',t)} = \langle \hat{\psi}_\alpha(\mathbf{r},t)\hat{\psi}_{-\alpha}(\mathbf{r}',t)\rangle.$$

Heisenberg equations of motion take the form

$$i\hbar\frac{\partial\hat{\psi}_{\uparrow,\downarrow}(\mathbf{r},t)}{\partial t} = [\mathcal{H}_0-\mu]\,\hat{\psi}_{\uparrow,\downarrow}(\mathbf{r},t) + \int W_{\downarrow,\uparrow}(\mathbf{r},\mathbf{r}',t)\hat{\psi}_{\uparrow,\downarrow}(\mathbf{r}',t)d^3r' \pm \int \Delta(\mathbf{r},\mathbf{r}',t)\hat{\psi}^\dagger_{\downarrow,\uparrow}(\mathbf{r}',t)d^3r',$$

where

$$W_\alpha(\mathbf{r},\mathbf{r}',t) = \int V\left(\mathbf{r}'-\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\mathbf{r}'+\frac{\mathbf{s}}{2}\right)\boxed{\bar{W}_\alpha\left(\mathbf{r}'-\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2},t\right)}\,d^3s,$$

and

$$\Delta(\mathbf{r},\mathbf{r}',t) = \int V(\mathbf{s},\mathbf{r}-\mathbf{r}')\boxed{\bar{\Delta}_\downarrow\left(\frac{\mathbf{r}'+\mathbf{r}-\mathbf{s}}{2},\frac{\mathbf{r}'+\mathbf{r}+\mathbf{s}}{2},t\right)}\,d^3s.$$

Quasi-homogeneous approximation

- BCS correlation length $\gg$ range of the collisional potential.
- Equal population densities of two states,
- Hartree field $\boxed{U(\mathbf{r})} \equiv g\langle \psi^\dagger_{\uparrow\downarrow}(\mathbf{r})\psi_{\uparrow\downarrow}(\mathbf{r})\rangle$
- Pair function $\boxed{\Delta(\mathbf{r})} \equiv g\langle \psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$
- Using separable form, mean-fields become

$$W_\alpha(\mathbf{r},\mathbf{r}',t) \approx \boxed{U(\mathbf{r},t)} \int F\left(\mathbf{r}' - \mathbf{r} + \frac{\mathbf{s}}{2}\right) F^*\left(\mathbf{r} - \mathbf{r}' + \frac{\mathbf{s}}{2}\right) d^3s$$

$$\Delta_\alpha(\mathbf{r},\mathbf{r}',t) \approx \boxed{\Delta(\mathbf{r},t)}\, F^*(\mathbf{r} - \mathbf{r}')$$

Bogoliubov Quasiparticle Representation

$$\psi_\uparrow(\mathbf{r},t) = \sum_{\mathbf{k}} \left[u_{\mathbf{k}}(\mathbf{r},t)\gamma_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}}(\mathbf{r},t)\gamma^\dagger_{\mathbf{k}\downarrow} \right]$$

$$\psi_\downarrow(\mathbf{r},t) = \sum_{\mathbf{k}} \left[u_{\mathbf{k}}(\mathbf{r},t)\gamma_{\mathbf{k}\downarrow} + v^*_{\mathbf{k}}(\mathbf{r},t)\gamma^\dagger_{\mathbf{k}\uparrow} \right]$$

Time-independent quasi-particle Fermion annihilation operators $\gamma_{\mathbf{k}\alpha}$

$$\langle \gamma^\dagger_{\mathbf{k}\alpha}\gamma_{\mathbf{k}'\beta}\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\beta}\,\bar{n}_{\mathbf{k}}$$

$$\langle \gamma_{\mathbf{k}\alpha}\gamma_{\mathbf{k}'\beta}\rangle = 0$$

Fermi-function: $\bar{n}_{\mathbf{k}} = \dfrac{1}{\exp(\epsilon_{\mathbf{k}}/k_B T) + 1}$

Time-dependent Bogoliubov de Gennes equations

- General form:

$$
\begin{aligned}
i\hbar\frac{\partial u_{\mathbf{k}}(\mathbf{r},t)}{\partial t} &= \left[\mathcal{H}_0-\mu\right]u_{\mathbf{k}}(\mathbf{r},t)+\int W(\mathbf{r},\mathbf{r}',t)u_{\mathbf{k}}(\mathbf{r}',t)d^3r'+\int \Delta(\mathbf{r},\mathbf{r}',t)v_{\mathbf{k}}(\mathbf{r}',t)d^3r' \\
i\hbar\frac{\partial v_{\mathbf{k}}(\mathbf{r},t)}{\partial t} &= -\left[\mathcal{H}_0-\mu\right]v_{\mathbf{k}}(\mathbf{r},t)-\int W(\mathbf{r},\mathbf{r}',t)v_{\mathbf{k}}(\mathbf{r}',t)d^3r'+\int \Delta^*(\mathbf{r},\mathbf{r}',t)u_{\mathbf{k}}(\mathbf{r}',t)d^3r'
\end{aligned}
$$

- Use appropriate separable potential forms for W, Δ collisional interaction potential
- Reproduce the usual ground state BCS equations.

Relation to Standard BCS Theory

Spatially homogeneous:

$$U(\mathbf{r},t) \to U, \qquad \Delta(\mathbf{r},t) \to \Delta,$$

$$u_{\mathbf{k}}(\mathbf{r}) \to a_0^{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad v_{\mathbf{k}}(\mathbf{r}) \to b_0^{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad |a_0^{\mathbf{k}}|^2 + |b_0^{\mathbf{k}}|^2 = \frac{1}{L^3}$$

Time-independent BCS equations:

$$\epsilon_{\mathbf{k}}\, a_0^{\mathbf{k}} = \left(E_{\mathbf{k}} - \mu + U \boxed{\left|f\,(\mathbf{k}/2)\right|^2}\right) a_0^{\mathbf{k}} + \Delta \boxed{f^*(\mathbf{k})}\, b_0^{\mathbf{k}}$$

$$\epsilon_{\mathbf{k}}\, b_0^{\mathbf{k}} = -\left(E_{\mathbf{k}} - \mu + U \boxed{\left|f\,(\mathbf{k}/2)\right|^2}\right) b_0^{\mathbf{k}} + \Delta^* \boxed{f(\mathbf{k})}\, a_0^{\mathbf{k}}$$

Solutions

Standard solution: $f(\mathbf{k}) = \Theta(k_c - |\mathbf{k}|)$

$$E_{\mathbf{k}} = \frac{\hbar^2 k^2}{2M} \qquad \varepsilon_{\mathbf{k}} = \sqrt{(E_{\mathbf{k}} - \mu + U)^2 + |\Delta|^2}$$

$$U = g \sum_{\mathbf{k}} \left[|a_0^{\mathbf{k}}|^2 \bar{n}_{\mathbf{k}} + |b_0^{\mathbf{k}}|^2 (1 - \bar{n}_{\mathbf{k}}) \right] \qquad a_0^{\mathbf{k}} = \eta_{\mathbf{k}} \Delta$$

$$\Delta = -g \sum_{\mathbf{k}} a_0^{\mathbf{k}} b_0^{\mathbf{k}*} (1 - 2\bar{n}_{\mathbf{k}}) \qquad b_0^{\mathbf{k}} = \eta_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - E_{\mathbf{k}} + \mu - U)$$

Renormalizable Solutions

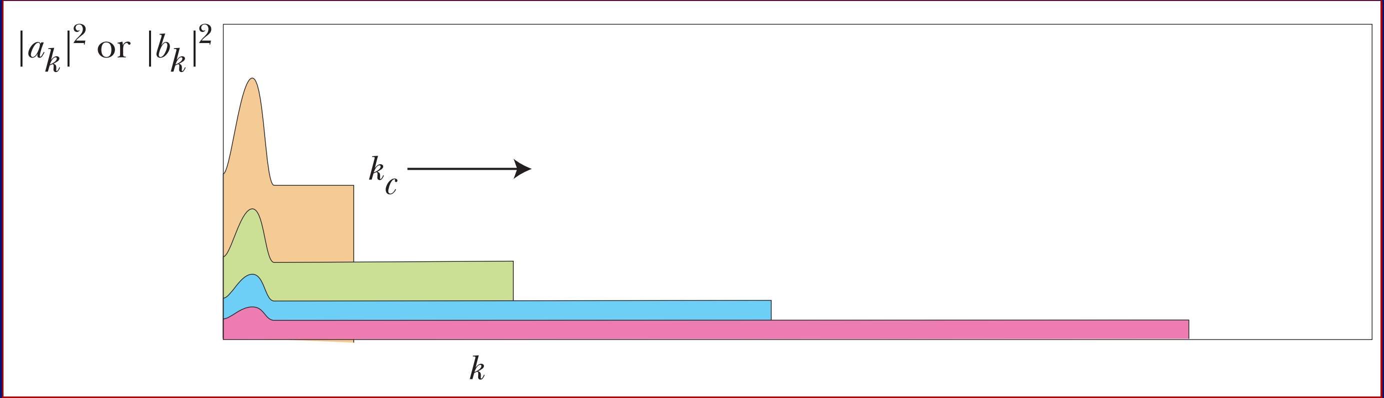
- For a ***given*** cutoff k_c, choose the strength g to give the correct scattering length.
- Same ***long wavelength*** physics independent of the cutoff.
- But as $k_c \to \infty$, $g \to 0$, and thus $U, \Delta \to 0$!

  BCS physics then expressed in terms of renormalized quantities

  $\tilde{\Delta} = T_{2B}(0)\Delta/g, \quad \tilde{U} = T_{2B}(0)U/g$
- Inclusion of optical ***Bragg potential*** $V_{\mathrm{opt}}(\mathbf{r}) \Longrightarrow$ ***cutoff dependence***

IS IT LEGAL TO HAVE A CUTOFF DEPENDENCE?

What Does Renormalization Do?



- Reduces a_k, b_k, while increasing the cutoff so that overall sums are preserved.
- Suppresses possibly interesting finite k behaviour in favour of indeterminate high k behaviour

Cutoff Dependence in the BCS Theory of Superconductivity

- Interaction between electrons mediated by crystal lattice
- Natural cutoff k_{Debye} given by Debye frequency
- Debye frequency depends on masses of crystal atoms
- Leads to the ***isotope effect***—critical temperature depends on the isotope of the superconducting metal.
- Experimentally well-verified effect
- Historically was evidence that the crystal lattice was involved in superconductivity
- Renormalization would ***destroy*** the isotope effect

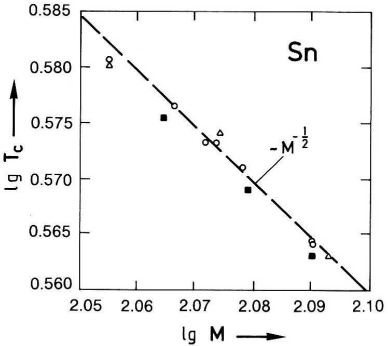


Fig. 10.15. Isotope effect for tin (Sn). The results of several authors are summarized [10.10]: Maxwell (○); Lock, Pippard, Shoenberg (■); Serin, Reynolds and Lohman (△)

4 Bragg Scattering Equations

Bloch form

$$
\begin{aligned}
u_{\mathbf{k}}(\mathbf{r},t) &= e^{i\mathbf{k}\cdot\mathbf{r}} \sum_n a_n^{\mathbf{k}}(t) e^{in(\mathbf{q}\cdot\mathbf{r}-\omega t)}, \\
v_{\mathbf{k}}(\mathbf{r},t) &= e^{i\mathbf{k}\cdot\mathbf{r}} \sum_n b_n^{\mathbf{k}}(t) e^{in(\mathbf{q}\cdot\mathbf{r}-\omega t)}, \\
U(\mathbf{r},t) &= \sum_n U_n(t) e^{in(\mathbf{q}\cdot\mathbf{r}-\omega t)}, \\
\Delta(\mathbf{r},t) &= \sum_n \Delta_n(t) e^{in(\mathbf{q}\cdot\mathbf{r}-\omega t)},
\end{aligned}
$$

Initial condition: amplitudes $u_{\mathbf{k}}(\mathbf{r},t)$ and $v_{\mathbf{k}}(\mathbf{r},t)$ are the BCS stationary states.

Evolution equations

$$
\begin{aligned}
i\hbar\frac{\partial a_n^{\mathbf{k}}(t)}{\partial t} &= \hbar\omega_n^a(\mathbf{k})a_n^{\mathbf{k}}(t) + \frac{1}{4}A\left[a_{n-1}^{\mathbf{k}}(t) + a_{n+1}^{\mathbf{k}}(t)\right] \\
&+ \sum_p \left\{U_{n-p}(t)|f\left(\tfrac{1}{2}(\mathbf{k}+p\mathbf{q})\right)|^2 a_p^{\mathbf{k}}(t) + \Delta_{n-p}(t)f^*(\mathbf{k}+p\mathbf{q})b_p^{\mathbf{k}}(t)\right\} \\
i\hbar\frac{\partial b_n^{\mathbf{k}}(t)}{\partial t} &= \hbar\omega_n^b(\mathbf{k})b_n^{\mathbf{k}}(t) - \frac{1}{4}A\left[b_{n-1}^{\mathbf{k}}(t) + b_{n+1}^{\mathbf{k}}(t)\right] \\
&- \sum_p \left\{U_{n-p}(t)|f\left(\tfrac{1}{2}(\mathbf{k}+p\mathbf{q})\right)|^2 b_p^{\mathbf{k}}(t) + \Delta_{p-n}^*(t)f(\mathbf{k}+p\mathbf{q})a_p^{\mathbf{k}}(t)\right\}
\end{aligned}
$$

where

$$
\hbar\omega_n^b(\mathbf{k}) = -E_{\mathbf{k}+n\mathbf{q}} + \mu - n\hbar\omega, \qquad \hbar\omega_n^a(\mathbf{k}) = E_{\mathbf{k}+n\mathbf{q}} - \mu - n\hbar\omega
$$

5 Numerical Results

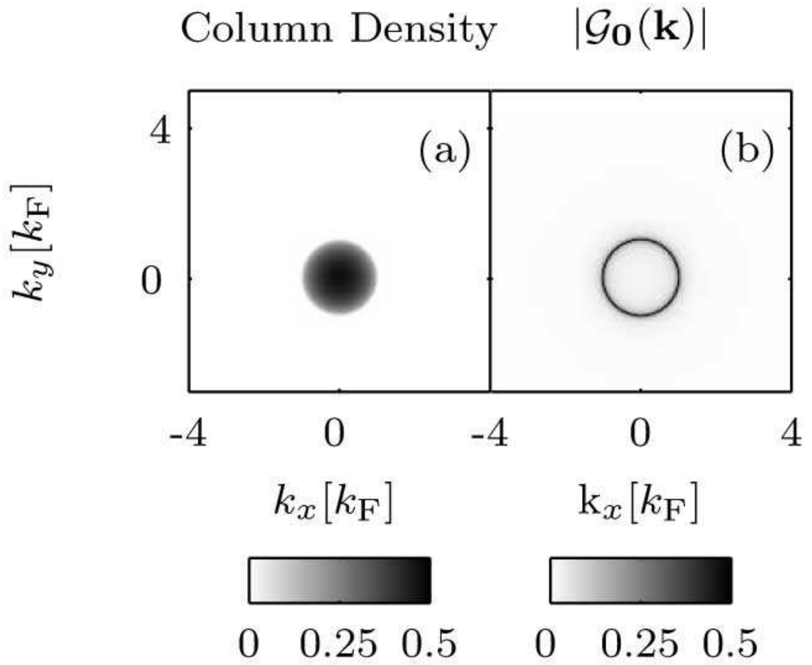
- Two-component gas
- Material ^{40}K in the $(F = 9/2, m_F = -9/2)$ and $(F = 9/2, m_F = -7/2)$ Zeeman states
- Cutoff $k_c \sim 31.1 k_F$.
- Scattering from initial pair correlations easily distinguished from the scattering in a non-interacting gas.

High momentum transfer

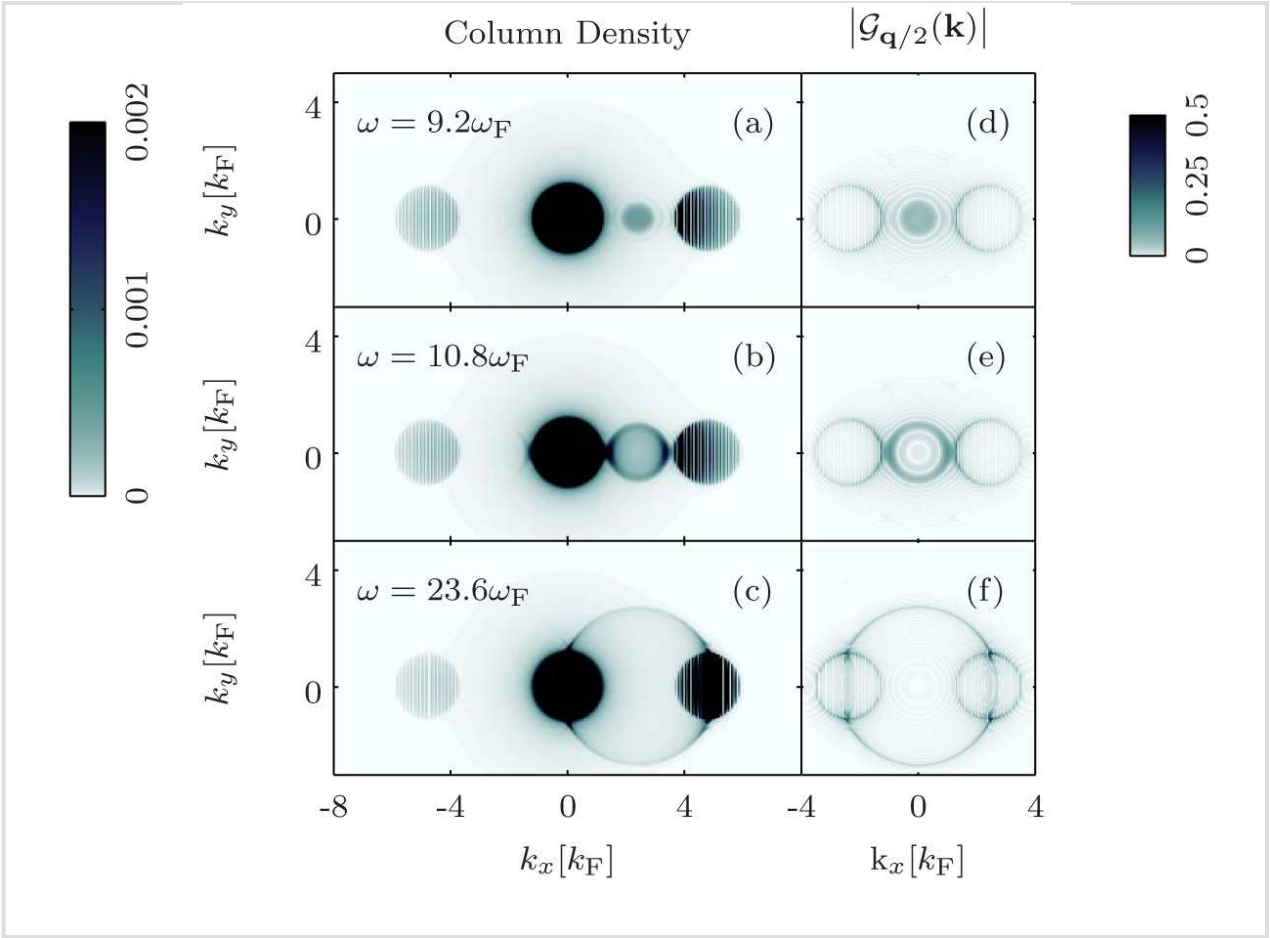
- $|\mathbf{q}| = 4.33 k_F \implies q/2$ well outside Fermi surface.
- 4 Bragg orders during the evolution

  Only $n = 0, \pm 1, -2$, for $a_n^{\mathbf{k}}$ and $b_n^{\mathbf{k}}$

Intial State—BCS Paired Fermi Gas



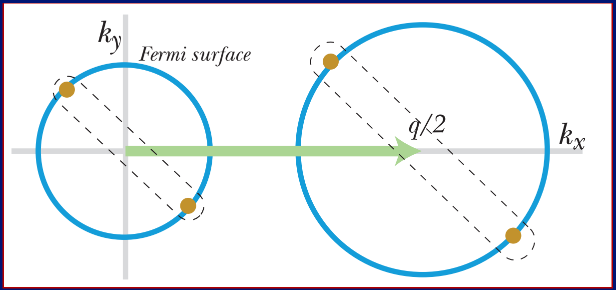
Bragg Scattering from Fermi Gas in BCS Regime



Interpretation—Correlated-pair Bragg scattering

- Possible in a Cooper paired Fermi gas because of the initial pair correlations.
- Characterized by
  - Spherical shell of correlated atoms in momentum space centred at momentum $\hbar\mathbf{q}/2$
  - On the red-detuned side of the single-particle Bragg resonance
  - Slight asymmetry in the Bragg spectra.
- Spherical shell not a direct result of single-particle Bragg scattering events
- Two stage process
  1. Generation of the pair potential grating via single-particle Bragg scattering
  2. Scattering of Cooper pairs via the Bragg grating in the pair potential.

Quantitative Model



- Momentum and energy conservation
- Zero COM Cooper pair—initial momenta $\pm\hbar\mathbf{k}_\mathrm{P}$.
- Mainly reside on the Fermi surface $\Longrightarrow |\mathbf{k}_\mathrm{P}| \sim k'_\mathrm{F}$.
- Cooper pair scattered by the pair potential grating to total COM momentum $\hbar\mathbf{q}$
- Excess energy distributed equally between the atoms of the pair—final momenta $\hbar(\mathbf{q}/2 \pm \mathbf{k}_\mathrm{rel})$.
- Energy required $\hbar\omega_\mathrm{pair} = \frac{\hbar^2}{M}\left(\frac{q^2}{4} + k_\mathrm{rel}^2 - k_\mathrm{F}'^2\right)$
- Frequency threshold, $\omega_\mathrm{thres} = \frac{\hbar}{M}\left(\frac{q^2}{4} - k_\mathrm{F}'^2\right)$,
- Below threshold frequency the pair potential grating does not have sufficient energy to scatter Cooper pairs
- Above the threshold scattered pairs form a spherical shell

  Momentum radius $k_\mathrm{rel} \approx \sqrt{\frac{M}{\hbar}\left(\omega - \omega_\mathrm{thres}\right)}$—Agrees well with detailed numerics.