Spin Hall Effect in Cold Atomic Systems

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Outline

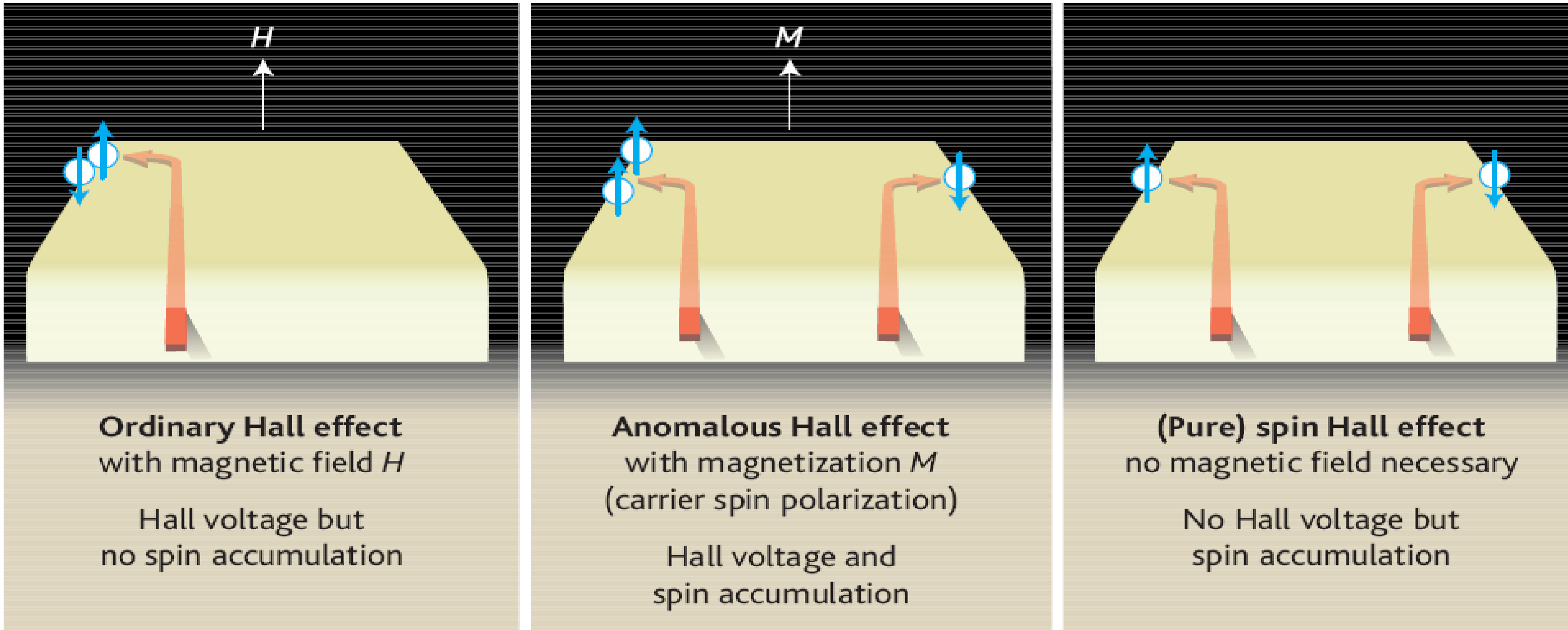
1. Fundamentals of Hall effect

- Comparing in AHE, QHE and SHE
- Spin Hall effect

2. Spin Hall effect in cold atomic gas

- Basic idea
- Quantum spin Hall effect in fermionic atomic gas
- Fractional spin Hall effect in cold bosonic atomic gas

1. Fundamentals of Hall Effect



(**Left**) The ordinary Hall effect is caused by deflection of carriers (electrons or holes) moving along an applied electric field by an applied magnetic field. Charge accumulation results in a Hall voltage, but there is no net spin accumulation because there are the same number of spin up carriers as spin down ones.

(**Middle**) The anomalous Hall effect is the result of spin-dependent deflection of carrier motion, which produces a Hall voltage and spin accumulation at the edges.

(**Right**) The pure spin Hall effect is caused by spin-dependent deflection of carriers and produces no Hall voltage when the numbers of deflected spin up and spin down electrons are the same but gives rise to spin accumulation. For simplicity, only the motion of a few carriers is shown in the figure panels.
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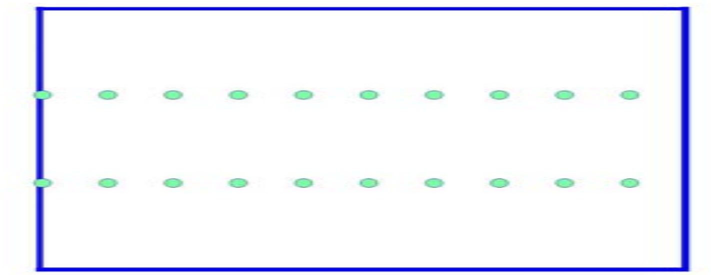
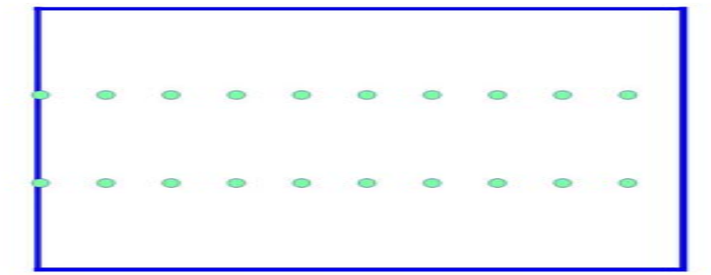
1.2 Spin Hall effect

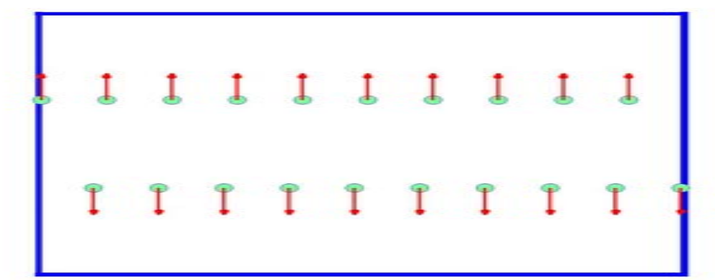
charge current: vector

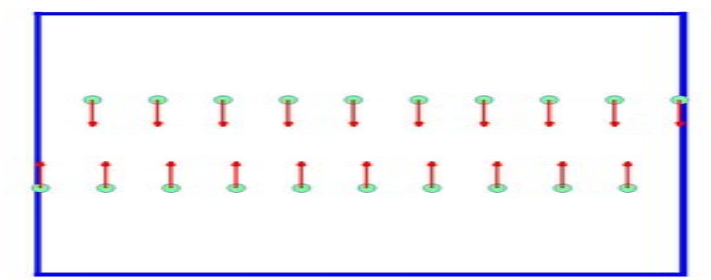




spin current: tensor

$$j_s^z = \frac{\hbar}{2}(S_z v_y + v_y S_z)$$



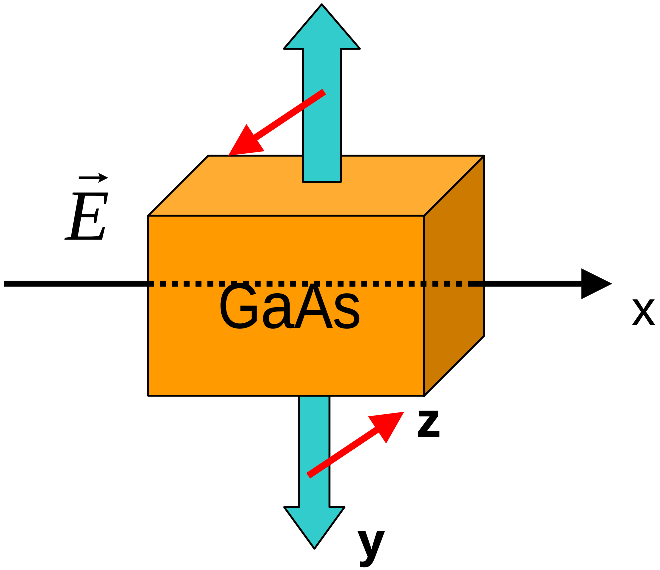


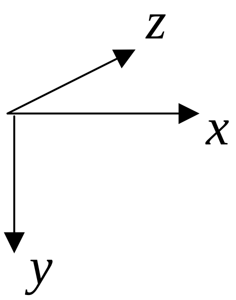
➢ Spin hall effect via electric field in semiconductors

a) **(3D) Luttinger Model**, **S. Murakami,N.Nagaosa, S-. C. Zhang, Science (2003);**

b) **(2D) Rashba Model**, **J. Sinova, et al, PRL (2004).**

In presence of spin-orbit coupling, the applied longitudinal electric field leads to a transverse motion of spins, with the spin-up and spin-down carriers transporting oppositely to each other, thus a spin current is created perpendicular to the electric field.



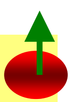


x: electric field
y: current direction
z: spin direction

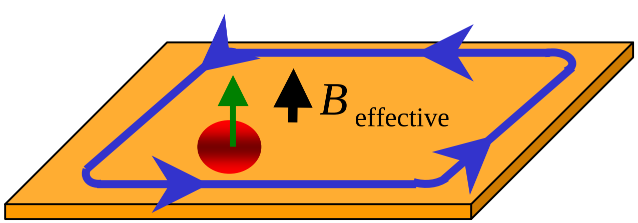
2. Spin Hall effect in atoms

➢ Quantum Spin hall effect?

By now theoretical studies on quantum spin Hall effect in solid systems are mainly included in **1) metallic graphene, 2) semiconductor system with strain gradients** and **3) HgTe/CdTe semiconductor quantum wells**. However, since the spin-orbit interaction in graphene is too small, the theoretical proposals in such systems are difficult to be achieved in experiment. Quantum spin Hall regime is also very difficult to achieve in semiconductor systems with strain gradients, due to the demanding requirement of a large strain gradient with special configuration and a very low electron density with a clean environment.

- Spin up 

$$\phi_n = \frac{z^n}{\sqrt{\pi n!}} e^{-\frac{1}{2}zz^*}$$

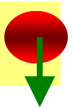


$$\sigma_{xy}^{charge\uparrow} = -\sigma_{xy}^{charge\downarrow} = \nu \frac{e^2}{h}$$

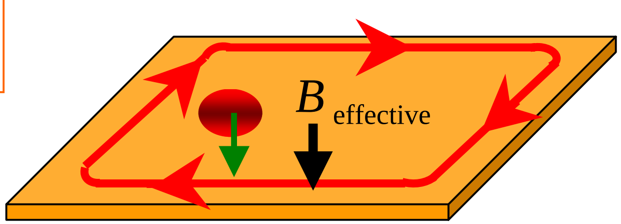
$$\sigma_{xy}^{charge} = 0;$$

$$\sigma_{xy}^{S\uparrow} = \sigma_{xy}^{S\downarrow} = \nu \frac{e}{4\pi}$$

$$\sigma_{xy}^{S} = \sigma_{xy}^{S\uparrow} + \sigma_{xy}^{S\downarrow} = 2\nu \frac{e}{4\pi};$$

- Spin down 

$$\phi_m = \frac{(z^*)^m}{\sqrt{\pi m!}} e^{-\frac{1}{2}zz^*}$$



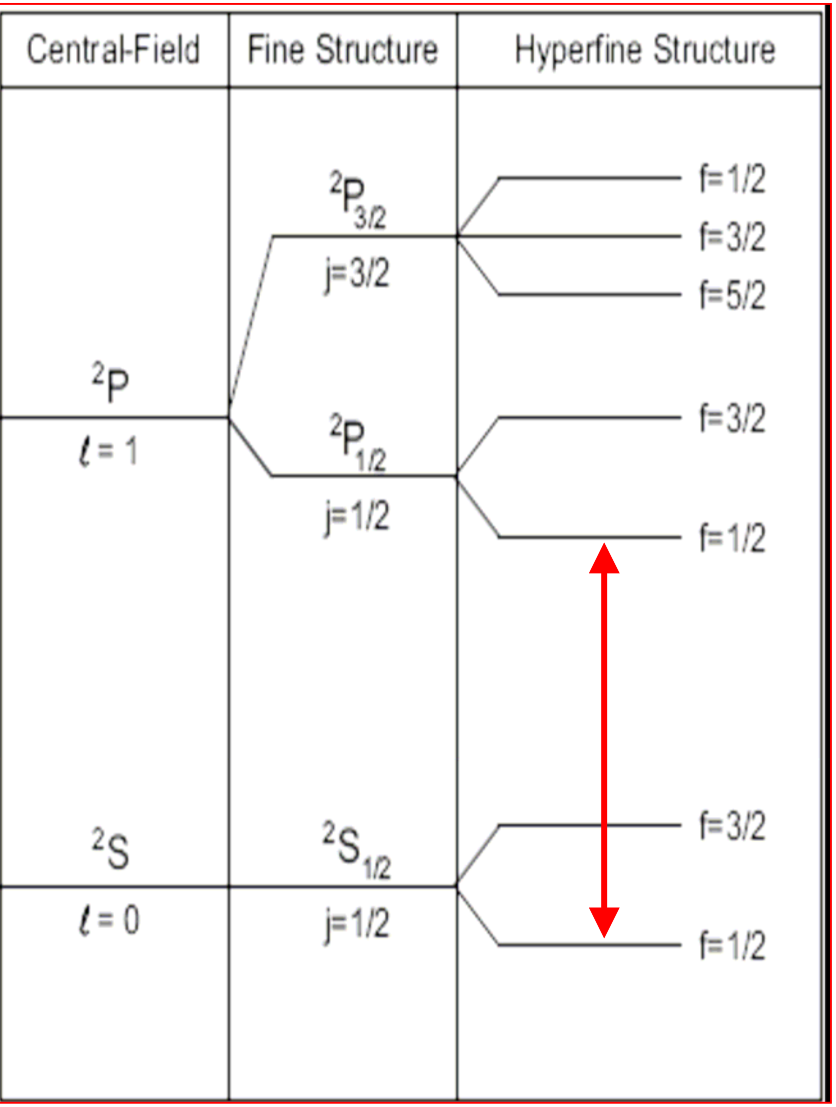
2.2 Quantum Spin hall effect in fermionic atomic gas

Electron: internal spin states, can be manipulated by magnetic field, spin-orbit coupling etc.

Atomic spin: internal angular-momentum states of atoms, can be manipulated by optical coupling.

Optical coherent control method provides a remarkable controllability in the dynamics of atomic spin states. Furthermore, parameters of cold atomic systems, e.g. atomic number, atom-atom interacting strength, can be well controlled in current experiments. This makes it possible to control the atomic spin propagation through optical methods, and further demonstrate the quantum spin hall effect (SHE) in neutral atomic system.

Model: A four-level system with 6Li atoms interacting with two angular momentum optical fields



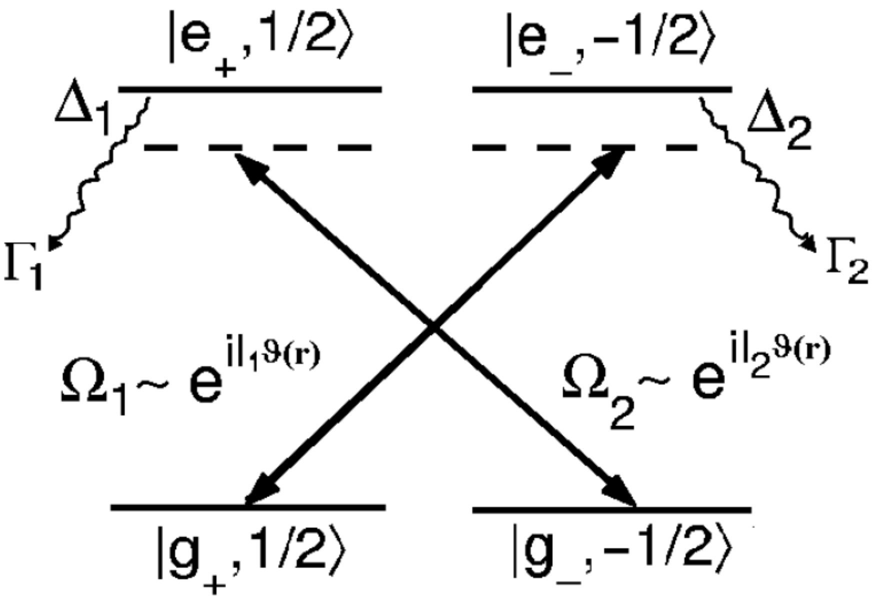


FIG. 1: Fermi atoms with four-level internal hyperfine spin states interacting with two light fields. This can be experimentally realized with alkali atoms, such as ^6Li atoms $(2^2S_{1/2}(F = 1/2) \longleftrightarrow 2^2P_{1/2}(F' = 1/2))$ [13].

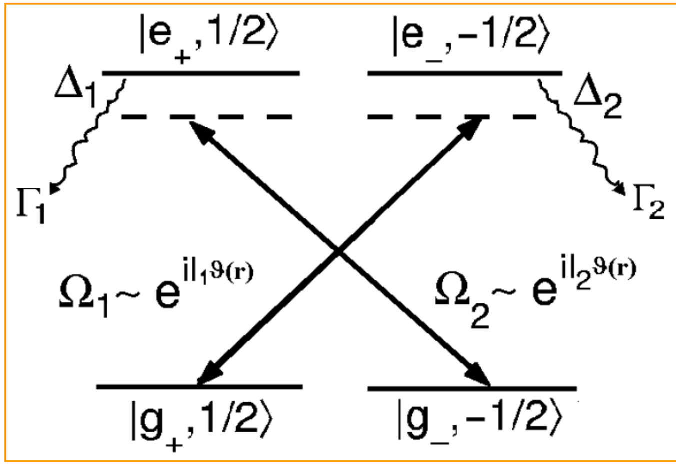
Hamiltonian $H = H_0 + H_1 + H_2$

$$
\begin{aligned}
H_0 &= \sum_{\alpha=e_\pm g_\pm} \int d^3r \Psi^*_\alpha \left(-\frac{\hbar^2}{2m}\nabla^2 + V_\alpha(\mathbf{r})\right)\Psi_\alpha, \\
H_1 &= \hbar\Delta_1 \int d^3r \Psi^*_{e_+} S_{e_+e_+} \Psi_{e_+} \\
&\quad + \hbar \int d^3r (\Psi^*_{e_+} \Omega_{10} e^{il_1\vartheta} S_{1+} \Psi_{g_-} + h.a.), \qquad (1) \\
H_2 &= \hbar\Delta_2 \int d^3r \Psi^*_{e_-} S_{e_-e_-} \Psi_{e_-} \\
&\quad + \hbar \int d^3r (\Psi^*_{e_-} \Omega_{20} e^{il_2\vartheta} S_{2+} \Psi_{g_+} + h.a.)
\end{aligned}
$$

with the atomic operators $S_{e_+e_+} = |e_+\rangle\langle e_+|$, $S_{e_-e_-} = |e_-\rangle\langle e_-|$, $S_{1+} = |e_+\rangle\langle g_-|$, $S_{2+} = |e_-\rangle\langle g_+|$, $S^\dagger_{1+} = S_{1-}$ and $S^\dagger_{2+} = S_{2-}$. The collisions (s-wave scattering) between cold Fermi atoms are negligible.

Diagonalization of the interaction Hamiltonian

$$\tilde{H}_I = U(\mathbf{r}) H_I U^\dagger(\mathbf{r}) = U(\mathbf{r})(H_1 + H_2)U^\dagger(\mathbf{r})$$



$$U(\mathbf{r}) = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

with

$$U_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j e^{il_j\vartheta(r)} \\ -\sin\theta_j e^{-il_j\vartheta(r)} & \cos\theta_j \end{bmatrix}, \qquad j = 1, 2$$

Then the four eigenstates read:

$$\begin{bmatrix} |\psi_+ > \\ |\psi_- > \end{bmatrix} = U_1 \begin{bmatrix} |e_+ > \\ |g_- > \end{bmatrix},$$

$$\begin{bmatrix} |\varphi_+ > \\ |\varphi_- > \end{bmatrix} = U_2 \begin{bmatrix} |e_- > \\ |g_+ > \end{bmatrix}$$

Large detuning condition: $\Delta_j^2 \gg \Omega_{j0}^2$

$$E^+_{\psi,\phi} \gg E^-_{\psi,\phi}$$

Under adiabatic condition:

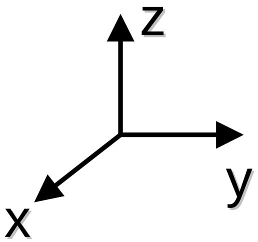
$$\boxed{|\Psi\rangle = \cos\gamma |S_\downarrow\rangle + \sin\gamma |S_\uparrow\rangle}$$

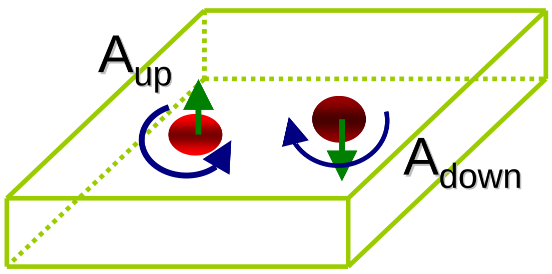
$$|S_\downarrow\rangle = |\psi_-\rangle, \; |S_\uparrow\rangle = |\phi_-\rangle$$

Effective Hamiltonian

Considering the large detuning case, the effective Hamiltonian :

$$
\begin{aligned}
H &= \int d^3 r \Psi^*_{s_\downarrow} \left[\frac{1}{2m}(\hbar\partial_k + i\frac{e}{c}A_k)^2\right] \Psi_{s_\downarrow} \\
&+ \int d^3 r \Psi^*_{s_\uparrow} \left[\frac{1}{2m}(\hbar\partial_k - i\frac{e}{c}A_k)^2\right] \Psi_{s_\uparrow} \\
&+ \int d^3 r (V_\downarrow |\Psi_{s_\downarrow}|^2 + V_\uparrow |\Psi_{s_\uparrow}|^2)),
\end{aligned}
$$





A Landau level structure for each spin orientation, essentially a direct analogy of SHE.

This is also equivalent to the system that an ensemble of two-flavor oppositely charged particles interact with one external effective gauge field.

Important features

1) Conserved spin current. Continuity equation:

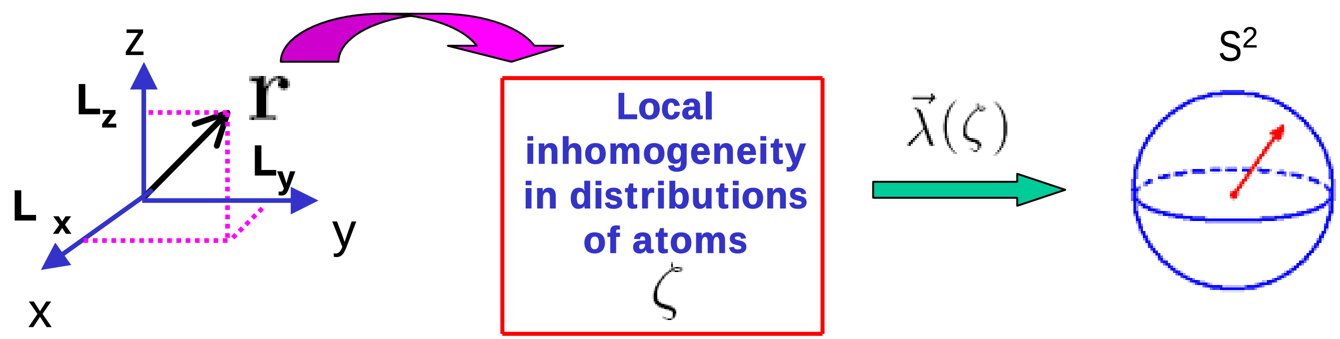
$$\begin{aligned}\vec{J}_k &= -\frac{i\hbar}{m}\vec{S}_{\downarrow}(\Psi^*_{s_\downarrow} D_{1k}\Psi_{s_\downarrow} - \Psi_{s_\downarrow} D^*_{1k}\Psi^*_{s_\downarrow}) \\ &\quad -\frac{i\hbar}{m}\vec{S}_{\uparrow}(\Psi^*_{s_\uparrow} D_{2k}\Psi_{s_\uparrow} - \Psi_{s_\uparrow} D^*_{2k}\Psi^*_{s_\uparrow})\end{aligned}$$

$$\boxed{\partial_t \vec{S}(\mathbf{r},t) + \partial_k \vec{J}_k = A_k \sigma_z \hat{\mathbf{e}}_z \times \vec{J}_k(\mathbf{r},t)}$$

$\longrightarrow$ $J_k^{s_z}$ conserved!

2) Topological properties

$$(\nabla \times \mathbf{J}^{s_z})_m = -2\hbar^2\eta^2\frac{e}{c}B_m - \frac{1}{2}\hbar^2\eta^2\epsilon_{mkl}\vec{\lambda}\cdot(\partial_k\vec{\lambda}\times\partial_l\vec{\lambda})$$



$$\boxed{\oint_s (\nabla \times \mathbf{J}^{s_z})\cdot d\mathbf{S} \sim \oint_s ds_m\epsilon_{mkl}\frac{\mathbf{r}}{r}\cdot(\partial_k\frac{\mathbf{r}}{r}\times\partial_l\frac{\mathbf{r}}{r}) \sim 4\pi n}$$

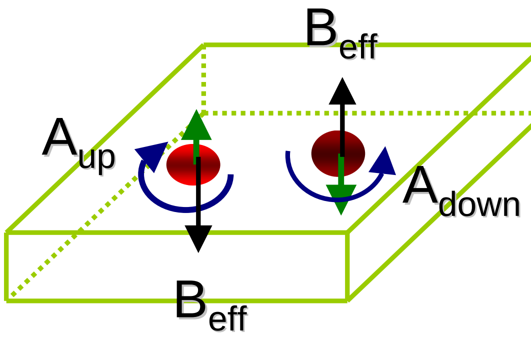
n: winding number

Uniform effective electric and magnetic field

For this we set the parameters:

$$\Omega_{01}(\mathbf{r}) = \Omega_{02}(\mathbf{r}) = f\rho$$

$$\boxed{\mathbf{B}_{\downarrow}(-\mathbf{B}_{\uparrow}) = \frac{\hbar l c}{e}\frac{f^2}{\Delta^2}\hat{e}_z, \mathbf{E} = -(1 + \frac{\hbar l^2 f^2}{4m\Delta^3})\frac{2\hbar f^2 x_0}{e\Delta}\hat{e}_x}$$



Pure spin current !

The spin / massive currents can be calculated with perturbation theory

$$J_{s_z}^y = n_a\left(\hbar + \frac{\hbar^2 l^2 f^2}{4m\Delta^3}\right)\frac{\Delta x_0}{l}, \qquad J_m = 0.$$

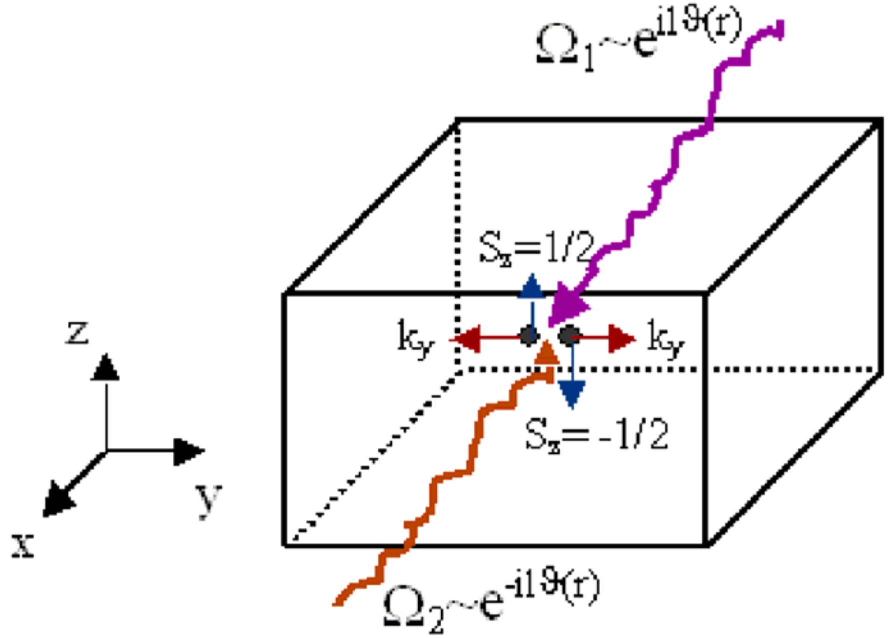


FIG. 2: (Color online) Under the interaction of effective electric and magnetic fields induced by light fields, atoms in state $|S_\downarrow\rangle$ ($S_z = -1/2$) and $|S_\uparrow\rangle$ ($S_z = 1/2$) have opposite momentums in y direction.

Quantization of spin Hall conductivity in optical lattice:

$$\sigma_{xy}^{charge\uparrow} = -\sigma_{xy}^{charge\downarrow} = \nu\frac{e^2}{h}$$

$$\sigma_{xy}^{charge} = 0;$$

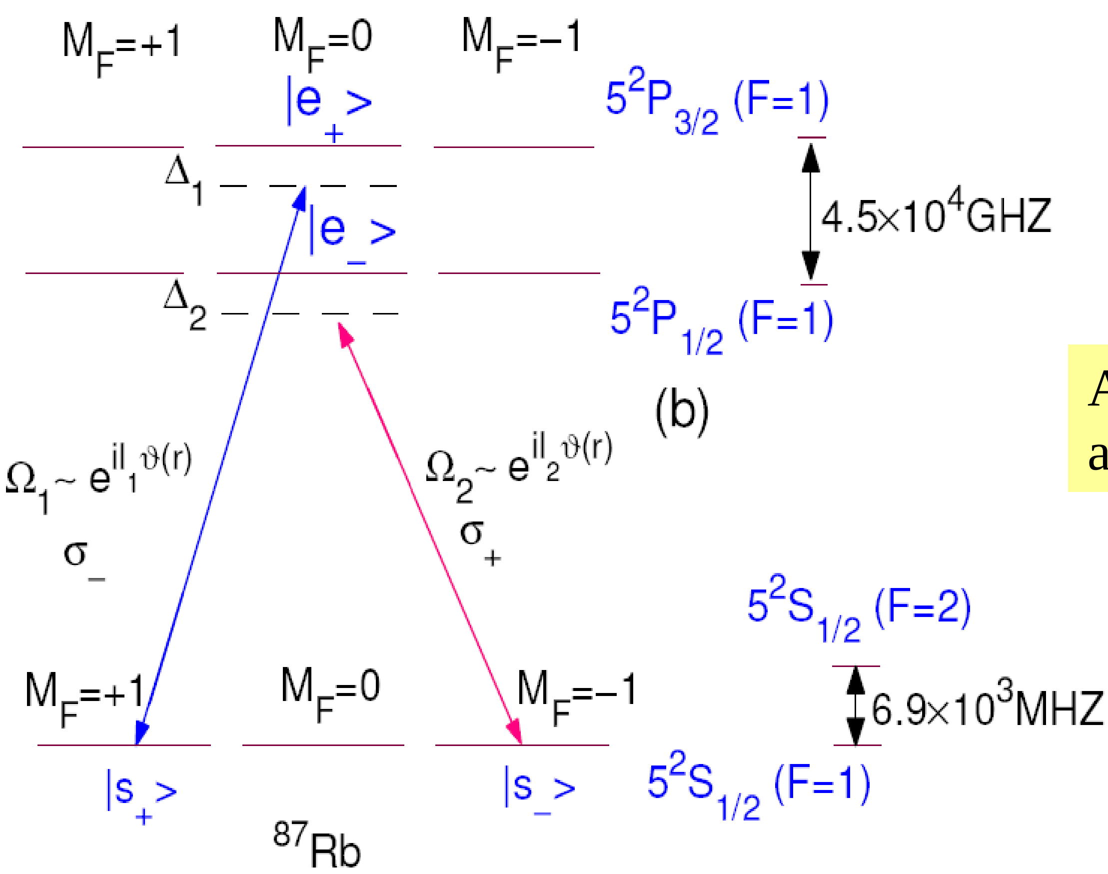
$$\sigma_{xy}^{S\uparrow} = \sigma_{xy}^{S\downarrow} = \nu\frac{e^2}{h}$$

$$\sigma_{xy}^{S} = \sigma_{xy}^{S\uparrow} + \sigma_{xy}^{S\downarrow} = 2\nu\frac{e^2}{h};$$

2.3 Fractional Spin Hall Effect

---from Fermionic system to Bosonic system

Realization of four-level bosonic system with 87Rb atoms



Similar to the former case, we consider here large detuning case:

$$|\Delta_{1,2}|^2 >> |\Omega_{1,2}|^2,$$

And the slowly-varying atomic field amplitudes are introduced by:

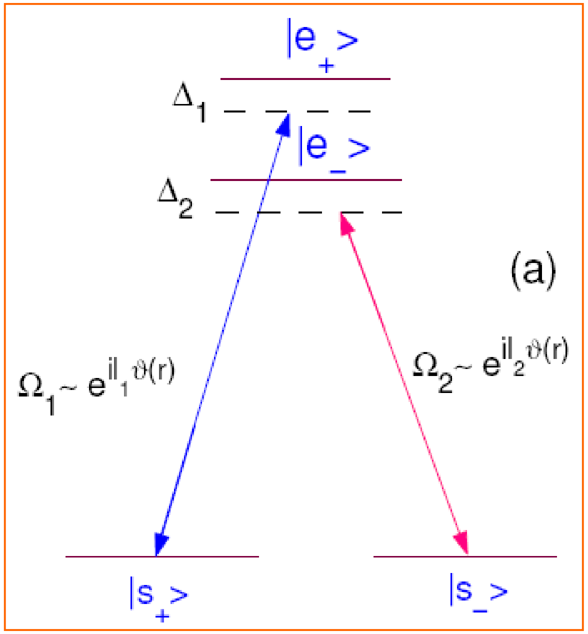
$$\phi_{s_\pm} = \Phi_{s_\pm}$$

$$\phi_{e_+} = \Phi_{e_+} e^{-i(\mathbf{k}_1\cdot\mathbf{r}-(\omega_{e_+}-\Delta_1)t)}$$

$$\phi_{e_-} = \Phi_{e_-} e^{-i(\mathbf{k}_2\cdot\mathbf{r}-(\omega_{e_-}-\Delta_2)t)}$$

Hamiltonian

Similar to the Fermionic system, the Hamiltonian of the present system is:



$$
\begin{aligned}
H_0 &= \sum_{\alpha=e_\pm, s_\pm} \int d^3r \phi^*_\alpha \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)\phi_\alpha + \\
&+ \sum_{\alpha,\beta} \int \underline{d^3r d^3r' \phi^*_\alpha(\mathbf{r})\phi^*_\beta(\mathbf{r}')U_{\alpha\beta}(\mathbf{r},\mathbf{r}')\phi_\alpha(\mathbf{r})\phi_\beta(\mathbf{r}')}, \\
H_1 &= \hbar\Delta_1 \int d^3r \phi^*_{e_+} S_{e_+e_+} \phi_{e_+} \\
&+\hbar \int d^3r (\phi^*_{e_+} \Omega_{10} e^{il_1\vartheta} S_{1+} \phi_{s_+} + h.a.), \\
H_2 &= \hbar\Delta_2 \int d^3r \phi^*_{e_-} S_{e_-e_-} \phi_{e_-} \\
&+\hbar \int d^3r (\phi^*_{e_-} \Omega_{20} e^{il_2\vartheta} S_{2+} \phi_{s_-} + h.a.),
\end{aligned}
$$

Atom-atom interaction part!

where the atom-atom scattering potential reads:

$$\boxed{U_{\alpha\beta}(\mathbf{r},\mathbf{r}') = 4\pi\hbar^2 a_{\alpha\beta}\delta^{(3)}(\mathbf{r}^\alpha - \mathbf{r}'^\beta)}$$

Quasi two-dimensional system

We can apply a tight harmonic confinement along z-axis with frequency ω_z such that z-axial ground state energy far exceeds any other transverse energy scale, yielding a quasi-2D system:

$$\boxed{\begin{aligned} H = & \sum_{j=1}^{N/2}\Big[\frac{1}{2m}\big(\mathbf{P}^{+j} - \frac{eB}{2c}\hat{e}_z \times \mathbf{r}_j^{+}\big)^2 + H_L^{+}\Big] \\ & + \sum_{j=1}^{N/2}\Big[\frac{1}{2m}\big(\mathbf{P}^{-j} + \frac{eB}{2c}\hat{e}_z \times \mathbf{r}_j^{-}\big)^2 + H_L^{-}\Big] \\ & + \sum_{j<k}\sum_{\alpha,\beta=+,-}\tilde{g}\,\delta^{(2)}(r_j^{\alpha} - r_k^{\beta}). \end{aligned}}$$

$$N_+ = N_- = N/2$$

Here B-field and angular-momentum part read:

$$B = \frac{\hbar l c}{e}\frac{f^2}{\Delta^2}\Big(1 + \frac{4m^2\Delta^2\omega_{eff}^2}{\hbar^2 l^2 f^4}\Big)^{\frac{1}{2}}$$

$$H_L^{\pm} = \pm(1-\Theta)eBL_z^{\pm}/4mc$$

And the 2D effective interacting constant:

$$\tilde{g} = a\sqrt{8\pi\hbar^3\omega_z^2/m}$$

Lowest-Landau-level (LLL) condition

To diagonalize the Hamiltonian and study the many-body Hamiltonian of this system, we shall consider the LLL condition. Energy value of the Hamiltonian is determined by three parts: (1) Landau Level; (2) angular-momentum part, $H_L^{\pm}$; (3) atom-atom interaction. Thus the LLL condition is determined by the following two requirements:

(A). The energy corresponding to angular momentum should be smaller than the interaction energy per particle:

$$\boxed{\varepsilon_{angular} << \varepsilon_{\mathrm{int}};}$$

(B). The interaction energy per particle should also be smaller than the spacing between Landau levels:

$$\boxed{\varepsilon_{\mathrm{int}} << \varepsilon_{landau};}$$

Under this condition, we can use the wave function of the lowest landau level to describe ground state of the present system.

Many-body function of the present system

The Hamiltonian can be diagonalized under LLL condition, and the many-body function of the ground state has the following form:

$$\Psi(z,\varpi^*) = Q(z,\varpi^*)\prod_{i<j}^{N/2}(z_i - z_j)^2\prod_{k<l}^{N/2}(\varpi_k^* - \varpi_l^*)^2 \times$$
$$\times\prod_{\mu,\nu}^{N/2}(z_\mu - \varpi_\nu^*)^w\prod_{j,k}^{N/2} e^{-|z_j|^2/2-|\varpi_k^*|^2/2}.$$

Here $w>0$ is a positive integer, and $z = x^+ + iy^+, \varpi = x^- + y^-$ are coordinates of spin-up and spin-down atoms, respectively. $Q(z,\varpi^*)$ is a homogeneous polynomial. One can verify the above an eigenstate of Hamiltonian with the eigenvalue:

$$E_L = (1-\Theta)eB(M_+ - M_-)/4mc.$$

Here $M_+>0$, $M_-<0$ are respectively the angular-momenta of all spin-up atoms and all spin-down atoms. As a result, the ground state of our system is determined by the angular momentum difference between spin-up and spin-down atoms.

Ground State I

The first type of ground state of the present system corresponds to $w=1$ and $\boxed{Q(z,\varpi^*)=1.}$

$$\boxed{\begin{aligned}\Psi(z,\varpi^*) &= \prod_{i<j}^{N/2}(z_i - z_j)^2 \prod_{k<l}^{N/2}(\varpi_k^* - \varpi_l^*)^2 \times \\ &\times \prod_{\mu,\nu}^{N/2}(z_\mu - \varpi_\nu^*) \prod_{j,k}^{N/2} e^{-|z_j|^2/2 - |\varpi_k^*|^2/2}.\end{aligned}}$$

Properties

The first type of ground state has the following properties:

- **Firstly, this state is analogous to the (221) type Halperin's function of two different spin states, but here the two spins experience opposite effective magnetic fields;**

- **Secondly, the 1st type of ground state is antisymmetric upon the interchange** $z \leftrightarrow \varpi^*$ **reflecting the** $|S_+>$ **chiral -** $|S_->$ **chiral *antisymmetry*;**

- **Thirdly, one can verify that the filling factor of type I ground state is:**

$$\boxed{\bar{v} = 1/3}$$

- **Finally, the angular momentum of spin-up or spin-down atoms or their total angular momentum is not conserved, but their difference is conserved!**

Ground State II

The second type of ground state of the present system corresponds to $w=2$ and $\boxed{Q(z,\varpi^{*})=1.}$

$$\boxed{\begin{aligned}\Psi(z,\varpi^{*}) &= \prod_{i<j}^{N/2}(z_i-z_j)^2\prod_{k<l}^{N/2}(\varpi_k^{*}-\varpi_l^{*})^2\times \\ &\times\prod_{\mu,\nu}^{N/2}(z_\mu-\varpi_\nu^{*})^2\prod_{j,k}^{N/2}e^{-|z_j|^2/2-|\varpi_k^{*}|^2/2}.\end{aligned}}$$

Properties

The second type of ground state has the following properties:

- **Firstly, this state is analogous to the (222) type Halperin's function of two different spin states, but here the two spins experience opposite effective magnetic fields;**

- **Secondly, the 2nd type of ground state is symmetric upon the interchange** $z \leftrightarrow \varpi^*$ **reflecting the** $|S_+>$ **chiral -** $|S_->$ **chiral *symmetry*;**

- **Thirdly, one can verify that the filling factor of the type II ground state is:**

$$\boxed{\bar{v} = 1/4}$$

Restrictions in the LLL condition

Energy scale corresponding to (1) Landau level spacing; (2) atom-atom interaction; and (3) angular-momentum part, $H_L^{\pm}$ are respectively obtained by and should satisfy the relation:

$$\boxed{\varepsilon_{landau} = \hbar eB/mc,} \gg \boxed{\varepsilon_{\text{int}} \approx n_{atom}\,\tilde{g};} \gg \boxed{\varepsilon_{angular} = E_L/N,}$$

The requirements of the LLL condition can then read:

$$\boxed{N << \min\{\frac{\Theta^2}{1-\Theta^2}\frac{\hbar^2}{\tilde{g}m}, \frac{64\bar{v}^2}{1-\Theta^2}\frac{\tilde{g}m}{\hbar^2}\}.}$$

From the inequality, our numerical result implies that for the strongly interacting boson atomic gas ($\boxed{\tilde{g} \sim \hbar^2 / m}$), the number of atoms can be as large as N~10$^{2\sim3}$ without violating the LLL condition, and for the weakly interacting case ($\boxed{\tilde{g} \sim 0.1\hbar^2 / m}$) this number is about N~ 10$^{1\sim2}$. We therefore expect the many body functions of the first type and the second type of ground states obtained here can be reached with a few cold bosonic atoms via optical method.

Summary

- Spin Hall effect can be demonstrated in atomic system through optical methods, where the created spin current is conserved and may be able to exhibit topological properties.

- Spin Hall effect in cold bosonic atomic gas has very different significance from that in cold fermionic atomic system. In bosonic system, we have obtained fractional spin Hall effect (FSHE), and have studied intriguing properties of many-body function of the cold atoms.

This report is based on the papers:

Xiong-Jun Liu, Xin Liu, L. C. Kwek and C. H. Oh, Phys. Rev. Lett. 98, 026602 (2007)
Xiong-Jun Liu, Xin Liu, L. C. Kwek and C. H. Oh, Submitted, cond-mat/0701506.

Thank you!