# *Quantum Phase Transitions in Optical Cavity QED*

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# HIGHER ORDER CORRECTIONS TO THE DICKE SUPERRADIANT PHASE TRANSITION

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Received 4 September 1973

The phase transition in the Dicke model for superradiance obtained by Hepp and Lieb and Wang and Hioe is modified by eliminating the rotating wave approximation.

# Outline

- Single-mode Dicke model
  - – equilibrium phase transition
  - – *T*=0 quantum phase transition
- Proposed realisation in optical cavity QED

  Dimer, Estienne, Parkins & Carmichael, PRA **75**, 013804 (2007)
  - – Raman transition scheme
  - – open system dynamics – non-equilibrium phase transition
  - – monitoring the system: cavity output field
  - – critical behaviour of quantum entanglement
- Other possibilities for effective spin systems

# Dicke Model

- *N* two-level atoms at fixed positions in a cavity of volume *V* (constant coupling strength)
- Inter-atomic separations large ⇒ neglect direct interactions between atoms

- However, the atoms interact with the same radiation field
  ⇒ they cannot be treated as independent, must be treated as a single quantum system

Dicke, Phys. Rev. **93**, 99 (1954)

# The Single-Mode Dicke Model

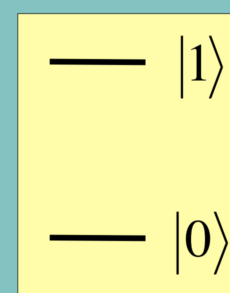
- *N* two-level atoms coupled identically to a single EM field mode

$$H_{\text{Dicke}} = \omega a^{+} a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}\left(a + a^{+}\right)\left(J^{-} + J^{+}\right)$$

- Coupling constant $\lambda \propto \sqrt{\frac{N}{V}}$

- Collective atomic operators

$$J^{-} = \sum_{i=1}^{N} |0_i\rangle\langle 1_i|, \quad J_z = \frac{1}{2}\sum_{i=1}^{N}\left(|1_i\rangle\langle 1_i| - |0_i\rangle\langle 0_i|\right)$$



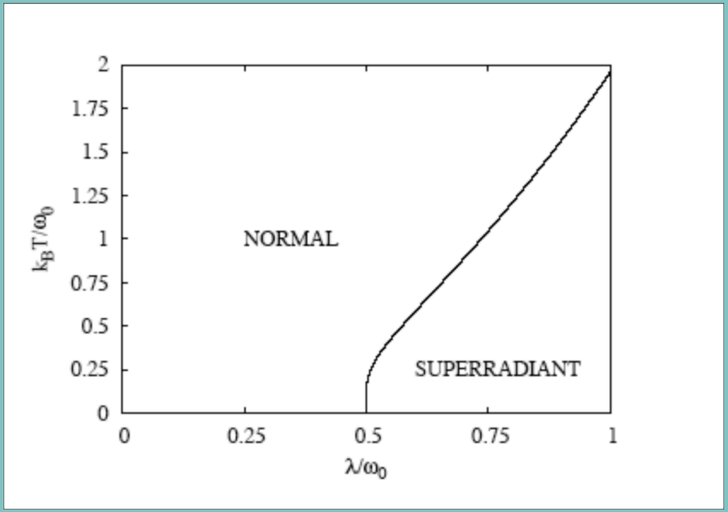
# Phase Transition in the Dicke Model

Hepp & Lieb, Phys. Rev. A **8**, 2517 (1973)
Hioe, Phys. Rev. A **8**, 1440 (1973)
Carmichael, Gardiner & Walls, Phys. Lett. **46A**, 47 (1973)

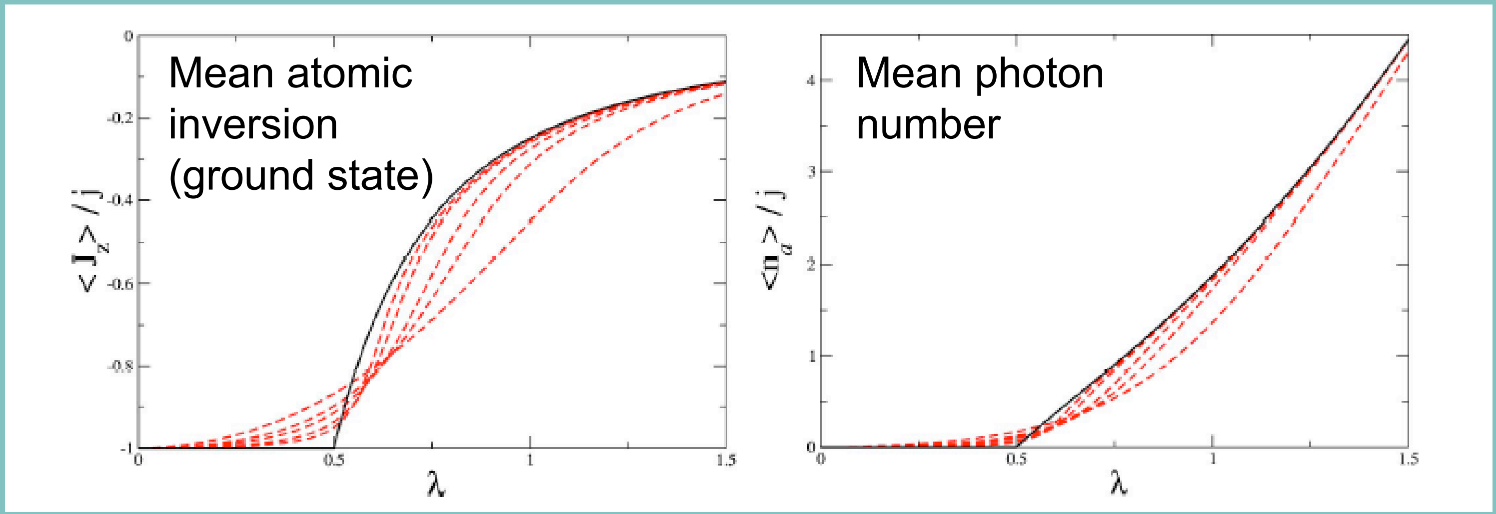
Thermodynamic limit
$N, V \to \infty, \quad N/V$ finite

- Phase transition to superradiant state for

$$\lambda > \lambda_c = \frac{\sqrt{\omega\omega_0}}{2}, \quad T < T_c \text{ where } \frac{\omega\omega_0}{4\lambda^2} = \tanh\left(\frac{\omega_0}{2k_B T_c}\right)$$



# “Order Parameters” ($T$=0)




(Dashed lines: finite atom number, $N$=1,2,3,6,10)

# But … no equilibrium phase transition with $A^2$ term included

## Phase Transitions, Two-Level Atoms, and the $A^2$ Term

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(Received 11 June 1975)

We show that the presence of the recently discovered phase transition in the Dicke Hamiltonian is due entirely to the absence of the $A^2$ terms from the interaction Hamiltonian.

Consider the well-studied Hamiltonian

$$H_1 = \frac{\hbar\omega_{ba}}{2}\sum_{j=1}^{N}\sigma_j^z + \hbar\omega a^\dagger a + \frac{\lambda}{\sqrt{N}}\sum_{j=1}^{N}(\sigma_j^+ a + \sigma_j^- a^\dagger). \qquad (1)$$

This Hamiltonian describes the collective interaction of a single mode of radiation (frequency $\omega$) with a single transition between levels $a$ and $b$ (frequency $\omega_{ba} > 0$) in $N$ identical two-level atoms. Operators $a$ and $a^\dagger$ denote here the annihilation and creation operators of the photons; $\sigma_j^z$, $\sigma_j^+$, $\sigma_j^-$ are Pauli matrices used to describe the $j$th atom. The Hamiltonian (1), sometimes called the Dicke Hamiltonian,[1] may be derived[2] from the more familiar one

$$H = \sum_{j=1}^{N}\left[\frac{1}{2m}\left(\vec{p}_j - \frac{e}{c}\vec{A}(\vec{r}_j)\right)^2 + V(\vec{r}_j)\right] + \hbar\omega a^\dagger a \qquad (2)$$

# Dicke Model Quantum Phase Transition ($T$=0)

Emary & Brandes, Phys. Rev. E **67**, 066203 (2003)

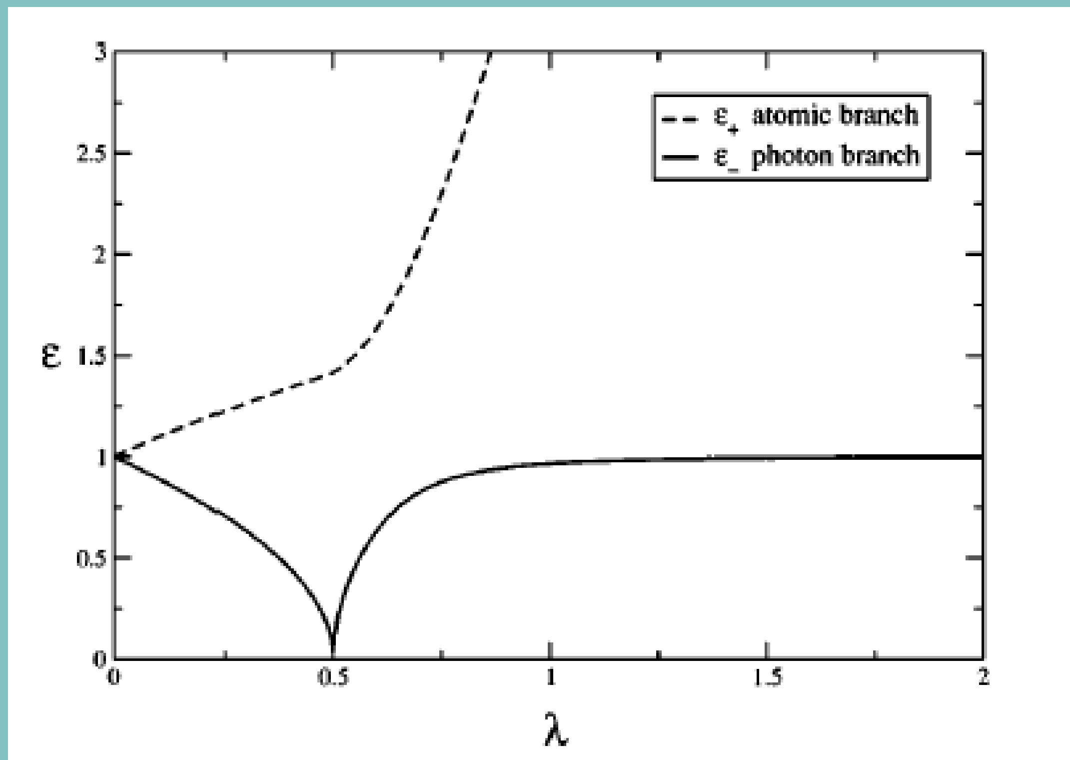
$$H_{\text{Dicke}} = \omega\, a^{+}a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}\left(a + a^{+}\right)\left(J^{-} + J^{+}\right)$$

- Holstein-Primakoff representation of angular momentum operators

$$J^{-} = \left(\sqrt{N - b^{+}b}\right) b, \quad J_z = b^{+}b - \frac{N}{2}, \quad \left[b, b^{+}\right] = 1$$

- Large-$N$ expansion of $H_{\text{Dicke}}$
  - → $H_{\text{normal}}$, $H_{\text{SR}}$ quadratic in ($a$,$a^+$,$b$,$b^+$)
  - → diagonalise (Bogoliubov transformation)
  - → excitation energies

# Excitation Energies $\left(\omega = \omega_0 = 1, \quad \lambda_c = 0.5\right)$



# Note: Derivation of $\{H_{\text{normal}}, H_{SR}\}$

$$\lambda < \lambda_c : \quad J^- \to \sqrt{N}\, b$$

$$\lambda > \lambda_c : \quad a \to a \pm \alpha, \quad b \to b \pm \beta \;\; \text{(coherent displacements)}$$

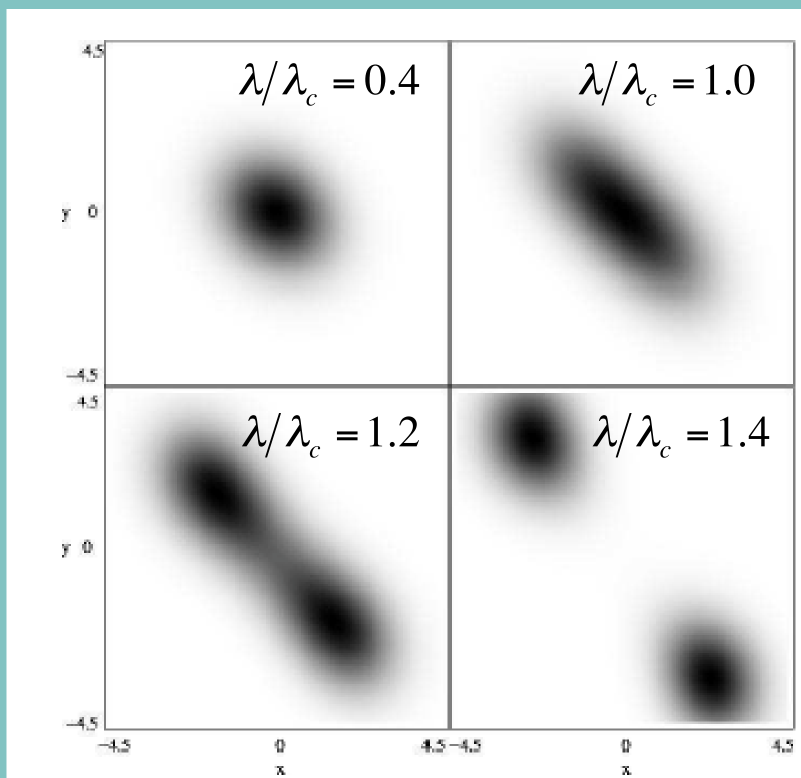
then expand in $N$

(i.e. linearisation about semiclassical amplitudes)

$$\langle a^+ a \rangle = |\alpha|^2, \quad \langle J_z \rangle = |\beta|^2 - \frac{N}{2}$$

# Note: Ground State "Wave Function"

$$|\psi(x,y)|^2$$ (*N* = 10 atoms)




Transition from localised state to delocalised "Schrödinger Cat" state

$$|\Psi_g\rangle \sim |\alpha\rangle|-N/2\rangle_x + |-\alpha\rangle|N/2\rangle_x$$

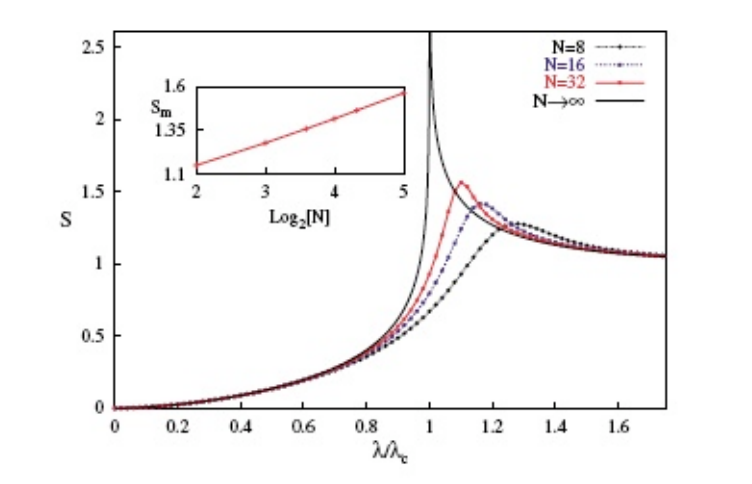
where

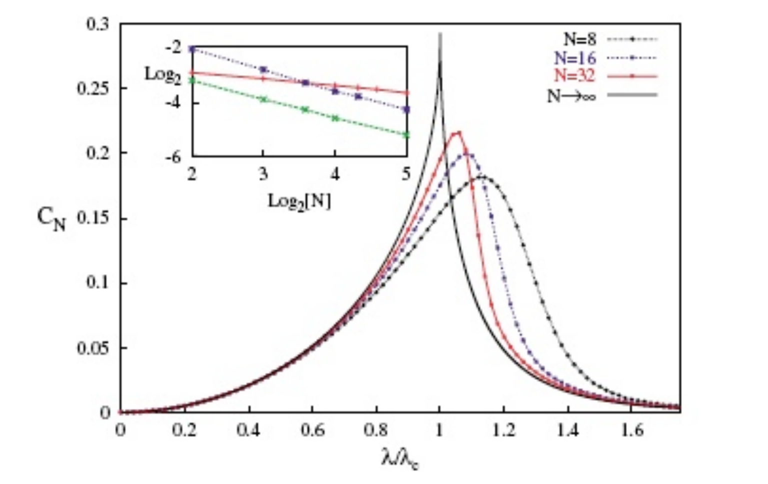
$$J_x|\pm N/2\rangle_x = \pm N/2|\pm N/2\rangle_x$$

# Entanglement properties

Critical behaviour of atom-field and atom-atom quantum entanglement at transition

Lambert, Emary & Brandes, Phys. Rev. Lett. **92**, 073602 (2004)



# Possible Realisation?

Issues to confront:

- To date, $\lambda << \{\omega, \omega_0\}$ in cavity QED experiments
- Atomic spontaneous emission, cavity mode losses
- And the $A^2$ issue

Our approach:

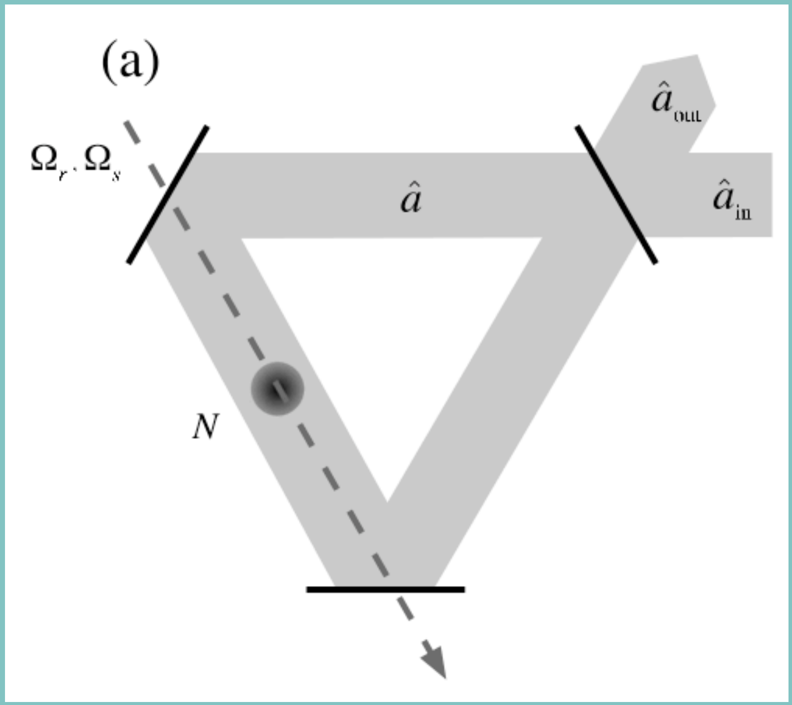
- Raman scheme, $\{\omega, \omega_0\} \propto$ {level shifts, Raman detunings},
  $\lambda \propto$ Raman transition rate
- Open-system dynamics

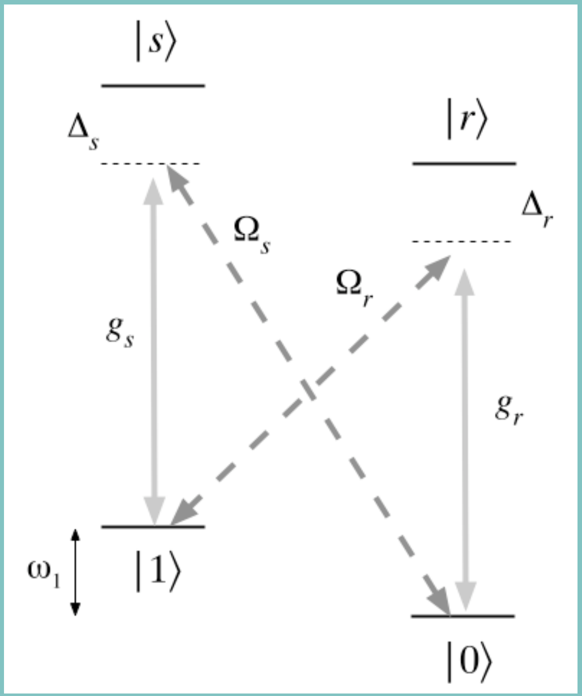
$\Rightarrow$ non-equilibrium (dynamical) quantum phase transition

Dimer, Estienne, Parkins & Carmichael, PRA **75**, 013804 (2007)

# Possible Realisation in Optical Cavity QED

- *N* atoms identically coupled to single optical (ring) cavity mode
- Lasers + cavity field drive two distinct Raman transitions between stable ground states |0> and |1>



# Model: Adiabatic elimination of atomic excited states

Effective Hamiltonian (rotating frame)

$$H = \left[\delta_{\text{cav}} + \frac{1}{2}N\left(\frac{g_r^2}{\Delta_r} + \frac{g_s^2}{\Delta_s}\right)\right]a^+a + \left(\frac{g_r^2}{\Delta_r} - \frac{g_s^2}{\Delta_s}\right)a^+aJ_z$$
$$+ \left(\frac{\Omega_r^2}{4\Delta_r} - \frac{\Omega_s^2}{4\Delta_s} + \delta'\right)J_z$$
$$+ \frac{g_r\Omega_r}{2\Delta_r}\left(aJ^+ + a^+J^-\right) + \frac{g_s\Omega_s}{2\Delta_s}\left(a^+J^+ + aJ^-\right)$$

$$\delta_{\text{cav}} = \omega_{\text{cav}} - \frac{1}{2}(\omega_{Ls} + \omega_{Lr})$$
$$\delta' = \omega_1 - \frac{1}{2}(\omega_{Ls} - \omega_{Lr})$$

Choose $\frac{g_s^2}{\Delta_s} = \frac{g_r^2}{\Delta_r}, \quad \frac{g_r\Omega_r}{2\Delta_r} = \frac{g_s\Omega_s}{2\Delta_s}$ then …

# Effective (Dissipative) Dicke Model

Master equation for atom-field density operator $\rho$ :

$$\dot{\rho} = -\mathrm{i}\left[H,\rho\right] + \kappa\left(2a\rho a^{+} - a^{+}a\rho - \rho a^{+}a\right)$$

where

$$H = \omega\, a^{+}a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}\left(a + a^{+}\right)\left(J^{-} + J^{+}\right)$$

with

$$\omega = \delta_{\mathrm{cav}} + \frac{N g_r^2}{\Delta_r}, \quad \omega_0 = \delta', \quad \lambda = \frac{\sqrt{N} g_r \Omega_r}{2\Delta_r}$$

"tunable" such that $\omega \sim \omega_0 \sim \lambda$

# Potential Experimental Parameters?

- Ring cavity / many atoms (e.g., Tübingen, Hamburg, $^{85}$Rb)

$$g_i/2\pi \approx 50\ \text{kHz}, \quad \kappa/2\pi \approx 20\ \text{kHz}, \quad N \approx 10^6$$

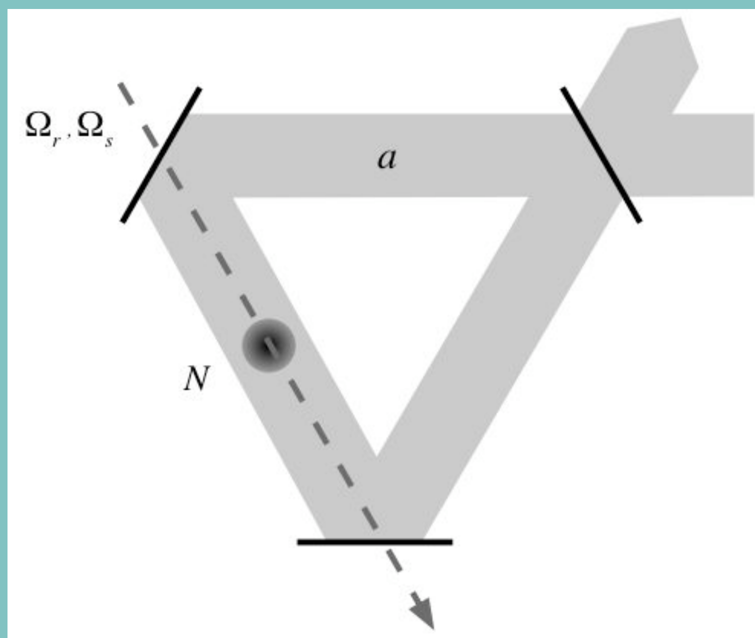
$$\frac{\Omega_i}{\Delta_i} \approx 0.005 \quad \Rightarrow \quad \frac{\lambda}{2\pi} \approx \sqrt{N} \times 0.125\ \text{kHz} \approx 125\ \text{kHz}$$

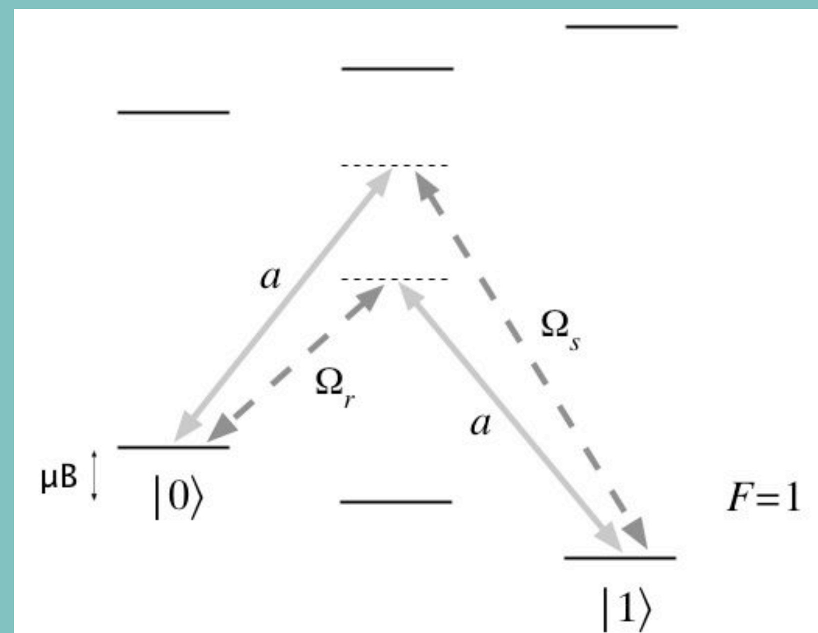
- Strong coupling CQED / few atoms (e.g., Georgia Tech, $^{87}$Rb)

$$g_i/2\pi \approx 30\ \text{MHz}, \quad \kappa/2\pi \approx 2\ \text{MHz}, \quad N \approx 100$$

$$\frac{\Omega_i}{\Delta_i} \approx 0.05 \quad \Rightarrow \quad \frac{\lambda}{2\pi} \approx \sqrt{N} \times 0.75\ \text{MHz} \approx 7.5\ \text{MHz}$$

e.g., ring cavity + $^{87}$Rb + magnetic field



# Holstein-Primakoff Analysis ($N \to \infty$): Normal Phase

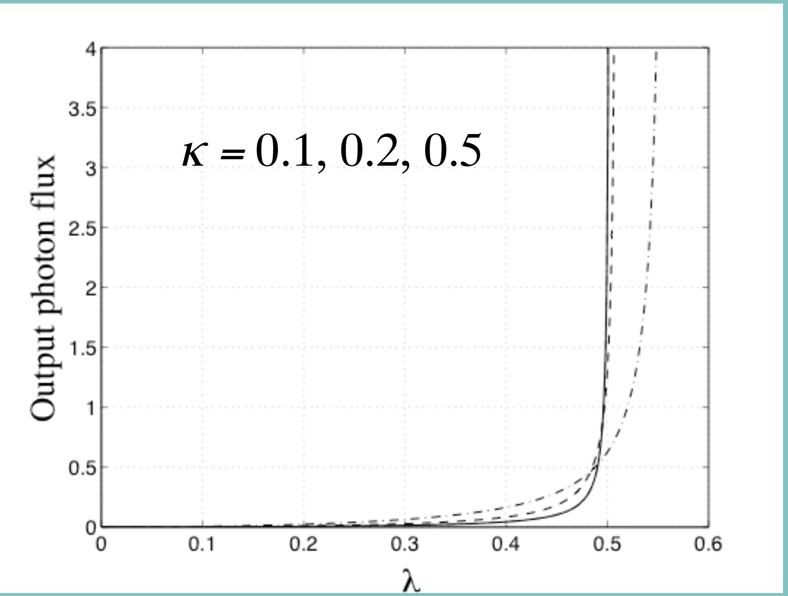
$$\dot{\rho} = -\mathrm{i}\left[H^{(1)}, \rho\right] + \kappa\left(2a\rho a^{+} - a^{+}a\rho - \rho a^{+}a\right)$$

with

$$H^{(1)} = \omega a^{+}a + \omega_0 b^{+}b + \lambda\left(a + a^{+}\right)\left(b + b^{+}\right)$$

for

$$\lambda < \lambda_c = \frac{1}{2}\sqrt{\frac{\omega_0}{\omega}\left(\kappa^2 + \omega^2\right)}$$



# Holstein-Primakoff Analysis ($N \to \infty$): Superradiant Phase

$$a \to c \pm \alpha, \quad b \to d \mp \beta$$

$$\dot{\rho} = -\mathrm{i}\left[H^{(2)}, \rho\right] + \kappa\left(2c\rho c^{+} - c^{+}c\rho - \rho c^{+}c\right)$$

with

$$H^{(2)} = \omega c^{+}c + \frac{\omega_0}{2\mu}(1+\mu)d^{+}d + \frac{\omega_0(1-\mu)(3+\mu)}{8\mu(1+\mu)}\left(d^{+}+d\right)^2$$

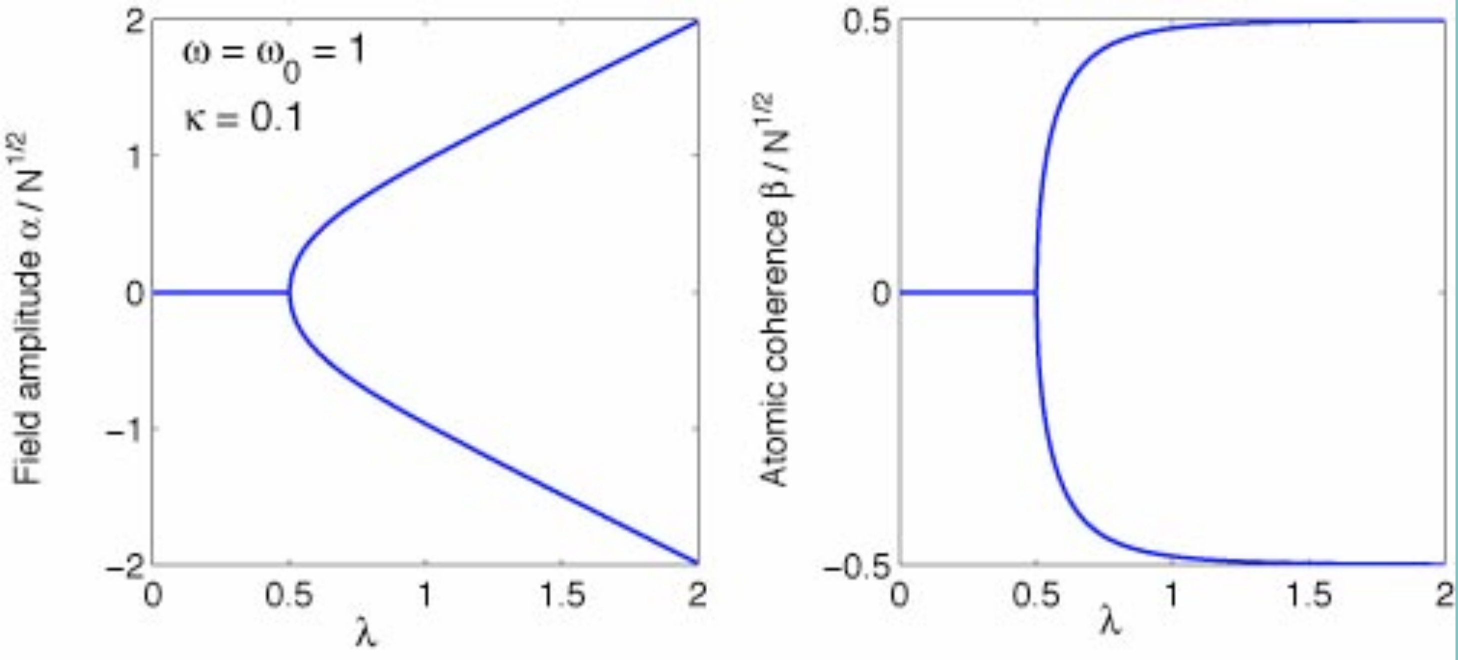
$$+\lambda\mu\sqrt{\frac{2}{1+\mu}}\left(c^{+}+c\right)\left(d^{+}+d\right), \qquad \mu = \frac{\lambda_c^2}{\lambda^2}$$

for

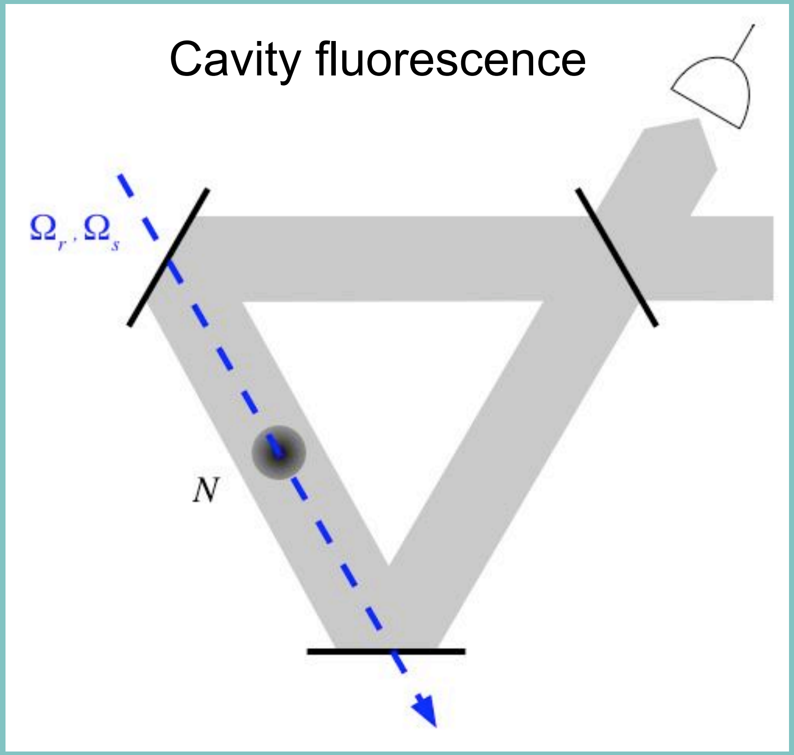
$$\lambda > \lambda_c = \frac{1}{2}\sqrt{\frac{\omega_0}{\omega}\left(\kappa^2+\omega^2\right)}$$

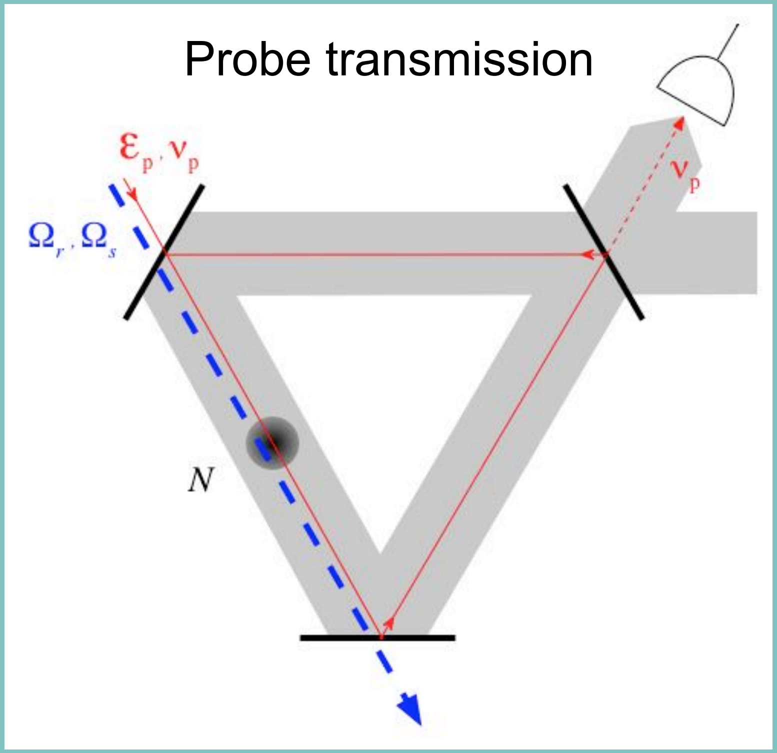
# Field and Atomic Amplitudes $\alpha$ and $\beta$

$$\alpha = \pm \frac{\lambda\omega_0}{2\lambda_c^2}\sqrt{\frac{N}{4}\left(1-\frac{\lambda_c^4}{\lambda^4}\right)}\left(1+\mathrm{i}\frac{\kappa}{\omega}\right), \qquad \beta = \mp\sqrt{\frac{N}{2}\left(1-\frac{\lambda_c^2}{\lambda^2}\right)}$$

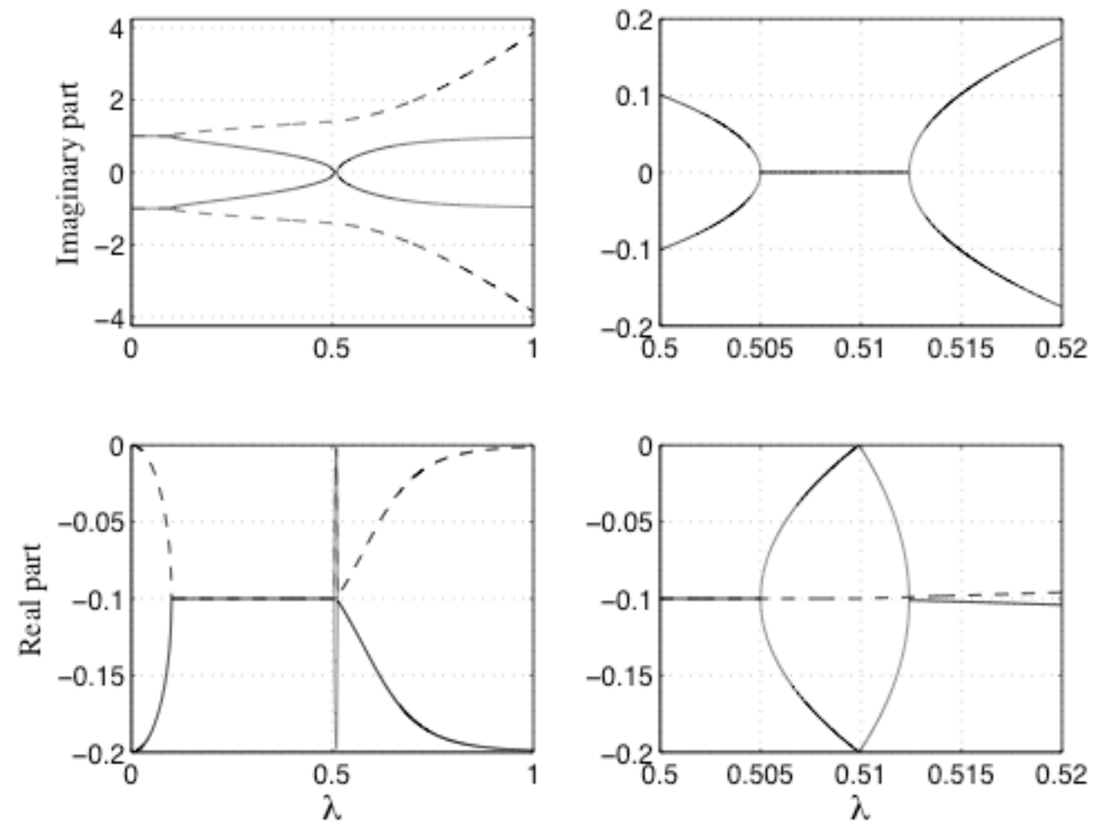


# Spectra of the Light Emitted from the Cavity

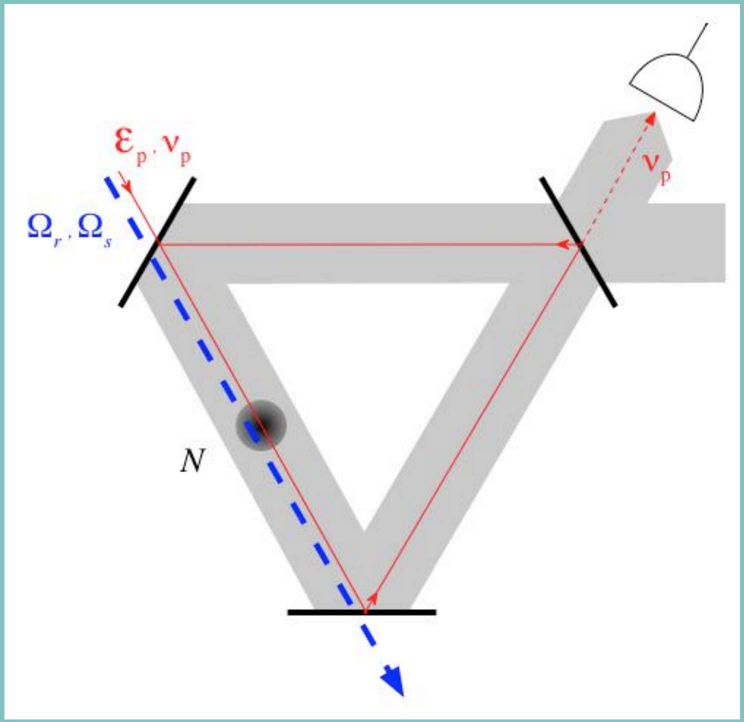


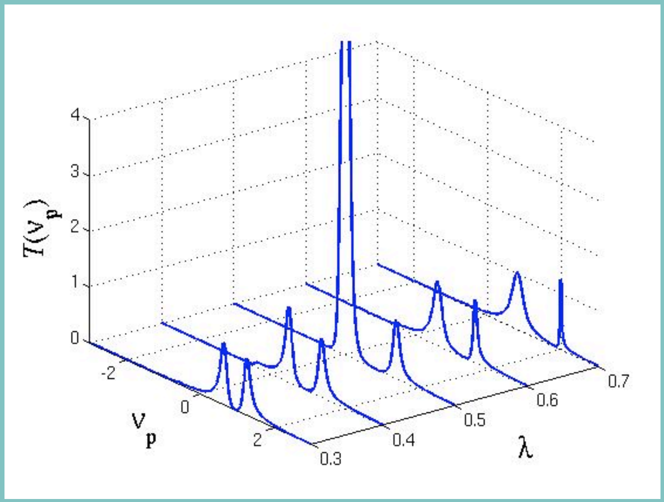



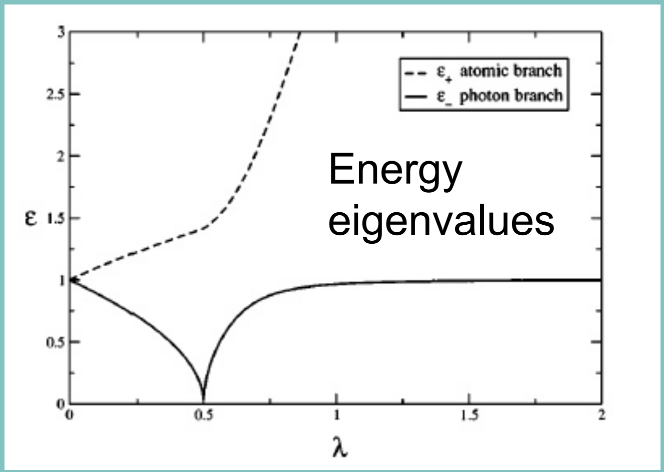
# Eigenvalues of the linearised model



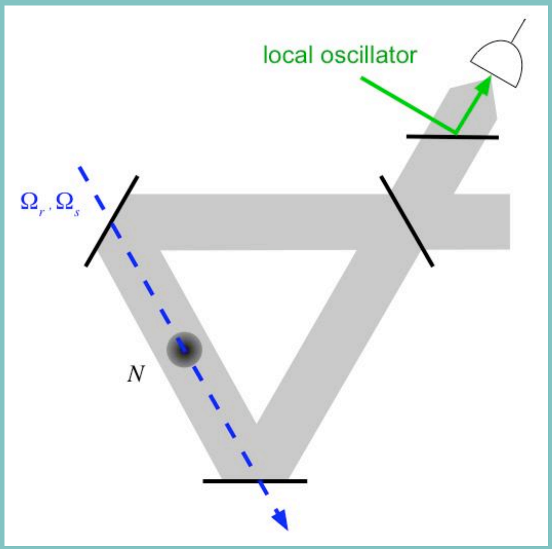
# Probe transmission spectra $(\omega = \omega_0 = 1, \kappa = 0.2)$

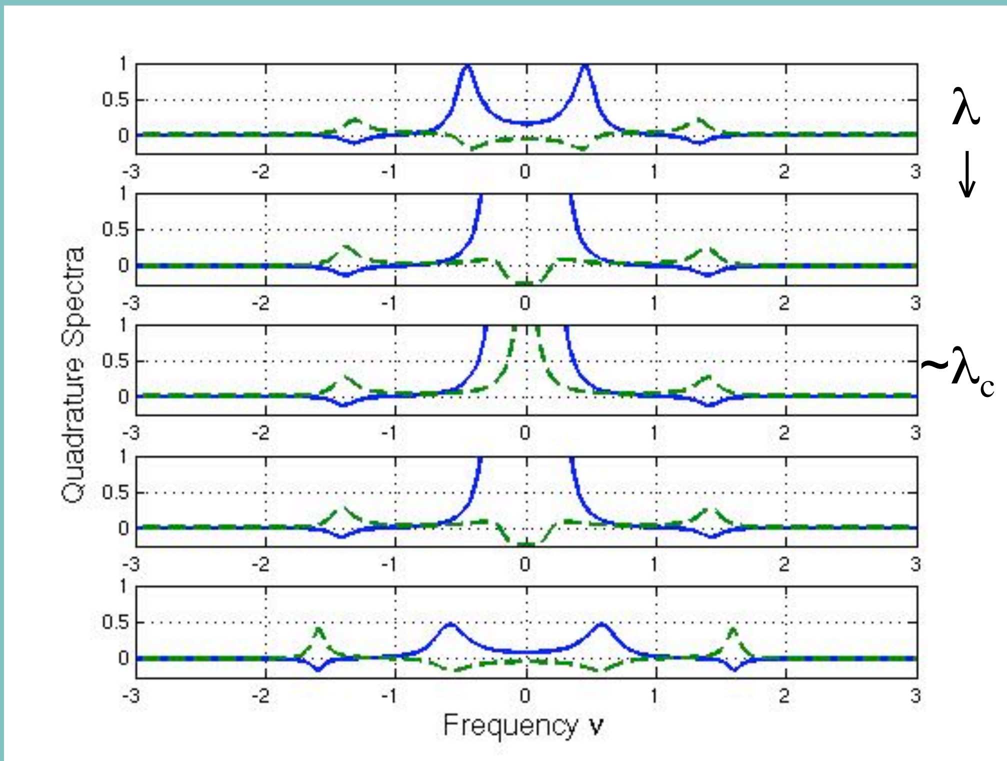


# Homodyne detection (quadrature fluctuation spectra)



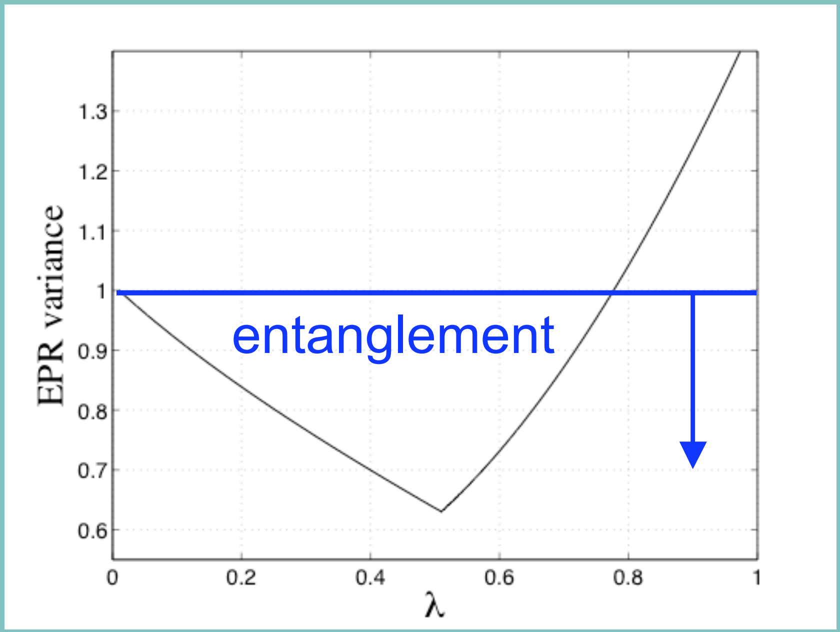



# Atom-field entanglement

Gaussian continuous variable state: quadrature/EPR operators

$$X_a^\theta = \frac{1}{2}\left(ae^{-i\theta} + a^+ e^{i\theta}\right), \quad X_b^\phi = \frac{1}{2}\left(be^{-i\phi} + b^+ e^{i\phi}\right)$$

$$u = X_a^\theta + X_b^\phi, \quad v = X_a^{\theta+\pi/2} - X_b^{\phi+\pi/2}$$




Possible to deduce from cavity output field

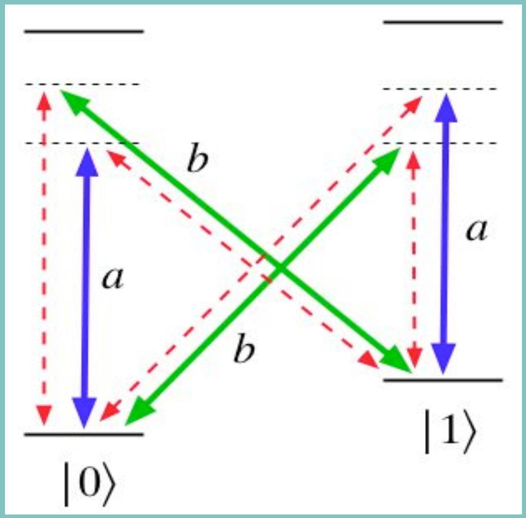
$$\left\langle (\Delta u)^2 \right\rangle + \left\langle (\Delta v)^2 \right\rangle < 1$$

# Other possibilities for effective spin systems

- Two cavity modes + off-resonant Raman transitions
- Effective spin-spin interactions:

$$H_{\text{eff}} = -2hJ_z - \frac{2\lambda}{N}\left(J_x^2 + \gamma J_y^2\right), \quad -1 < \gamma < 1$$

(Lipkin-Meshkov-Glick model)



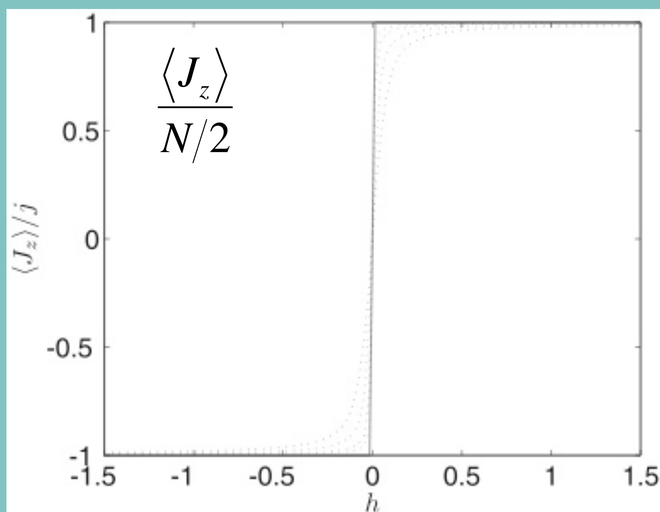

- $\lambda$ » dissipative rates possible
- 1st or 2nd order quantum phase transitions

Example:
("antiferromagnetic

$$H_{\text{eff}} = -2hJ_z - \frac{2\lambda}{N} J_x^2, \quad \underline{\lambda < 0}$$

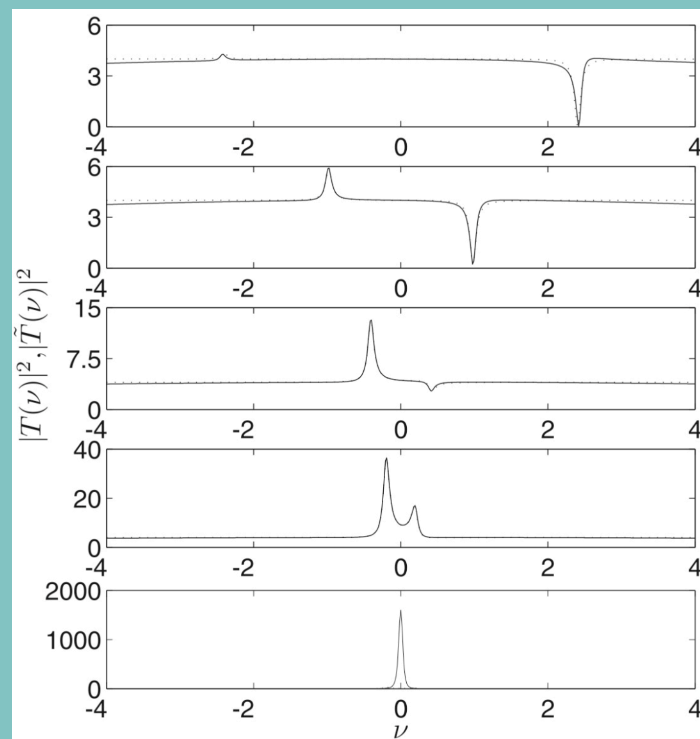
$$\dot{\rho} = -i\left[H_{\text{eff}}, \rho\right] + \frac{4\Gamma_a}{N}\left(2J_x\rho J_x - J_x^2\rho - \rho J_x^2\right) + \frac{\Gamma_b}{N}\left(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-\right)$$

1st-order quantum phase transition




$\lambda = -1,\ \Gamma_a = 0.01,\ \Gamma_b = 0.2$

Probe transmission spectrum

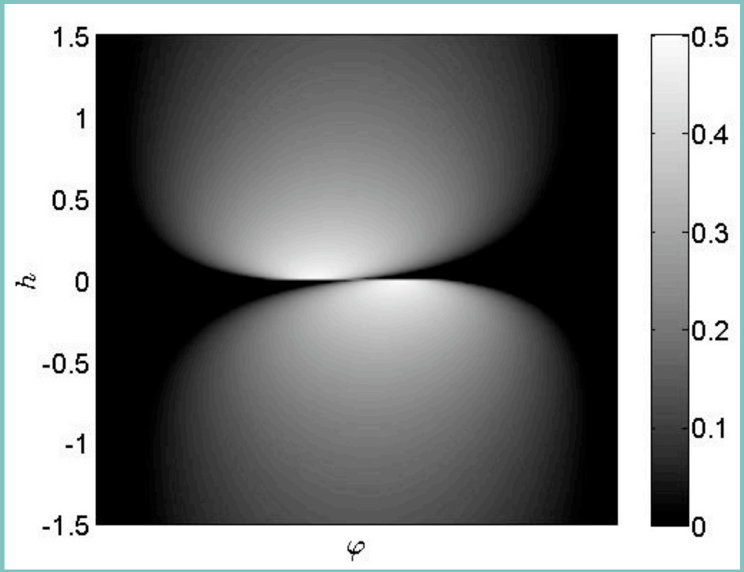



$h$=0.8

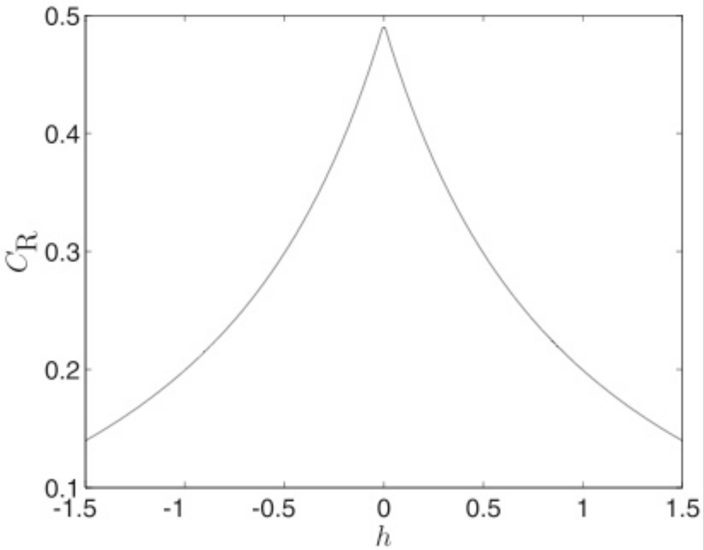
$h$=0.01

# Bipartite entanglement criterion / spin variances

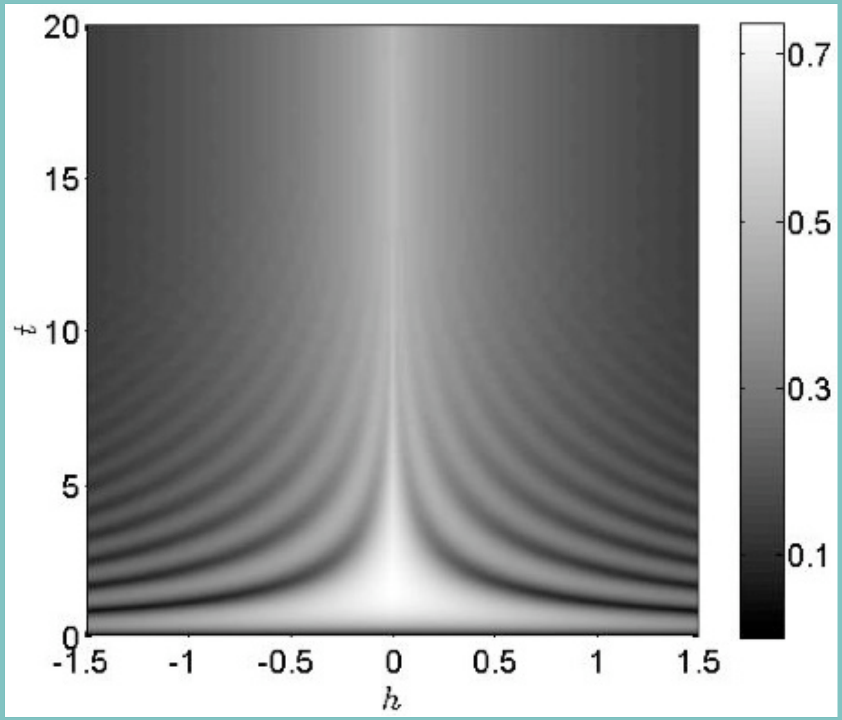
$$C_\varphi = 1 - \frac{4}{N}\left\langle \Delta J_\varphi^2 \right\rangle - \frac{4}{N^2}\left\langle J_\varphi \right\rangle^2 > 0, \quad J_\varphi = J_x \sin\varphi + J_y \cos\varphi$$




$$C_R = \max_\varphi C_\varphi$$



## Time-dependence of entanglement, $C_R(t)$



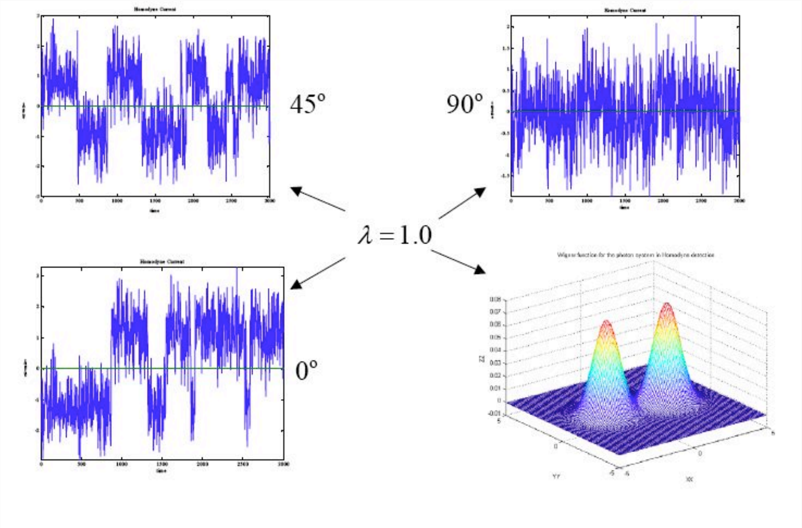

Note: Cavity output field $b_{out} \propto J_-$ so $C_R$ can be deduced from <u>measurable correlation functions</u>

# Summary

- Proposed realisation of Dicke model in cavity QED for study of (non-equilibrium) quantum phase transition
- Well-defined cavity output provides measurable signatures/properties of the phase transition
- Other effective spin models possible

# Further possibilities …

- Finite-*N* systems
  – small → large quantum noise
  – entangled state preparation
  and characterisation
  – measurement back-action




- Combination with optical-lattice many-body systems
  (long-range + short-range interactions)
- Disordered systems