Quantum Phase Transitions in Optical Cavity QED

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HIGHER ORDER CORRECTIONS
TO THE DICKE SUPERRADIANT PHASE TRANSITION

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The phase transition in the Dicke model for superradiance obtained by Hepp and Lieb and Wang and Hioe is modified by eliminating the rotating wave approximation.
Outline

- **Single-mode Dicke model**
  - equilibrium phase transition
  - $T=0$ quantum phase transition
- **Proposed realisation in optical cavity QED**
  Dimer, Estienne, Parkins & Carmichael, PRA 75, 013804 (2007)
  - Raman transition scheme
  - open system dynamics – non-equilibrium phase transition
  - monitoring the system: cavity output field
  - critical behaviour of quantum entanglement
- **Other possibilities for effective spin systems**
Dicke Model

- $N$ two-level atoms at fixed positions in a cavity of volume $V$ (constant coupling strength)
- Inter-atomic separations large $\Rightarrow$ neglect direct interactions between atoms
- However, the atoms interact with the same radiation field $\Rightarrow$ they cannot be treated as independent, must be treated as a single quantum system

Dicke, Phys. Rev. 93, 99 (1954)
The Single-Mode Dicke Model

- \( N \) two-level atoms coupled identically to a single EM field mode

\[
H_{\text{Dicke}} = \omega a^+ a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}} (a + a^+) \left( J^- + J^+ \right)
\]

- Coupling constant \( \lambda \propto \sqrt{\frac{N}{V}} \)

- Collective atomic operators

\[
J^- = \sum_{i=1}^{N} |0_i\rangle \langle 1_i|, \quad J_z = \frac{1}{2} \sum_{i=1}^{N} (|1_i\rangle \langle 1_i| - |0_i\rangle \langle 0_i|)
\]
Phase Transition in the Dicke Model

Hepp & Lieb, Phys. Rev. A 8, 2517 (1973)
Hioe, Phys. Rev. A 8, 1440 (1973)
Carmichael, Gardiner & Walls, Phys. Lett. 46A, 47 (1973)

- Phase transition to superradiant state for

\[
\lambda > \lambda_c = \sqrt{\omega \omega_0} \frac{\omega \omega_0}{2}, \quad T < T_c \quad \text{where} \quad \frac{\omega \omega_0}{4 \lambda^2} = \tanh \left( \frac{\omega_0}{2k_B T_c} \right)
\]

Thermodynamic limit
\[N, V \rightarrow \infty, \quad N/V \text{ finite}\]
“Order Parameters” ($T=0$)

(Dashed lines: finite atom number, $N=1,2,3,6,10$)
Phase Transitions, Two-Level Atoms, and the $A^2$ Term

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We show that the presence of the recently discovered phase transition in the Dicke Hamiltonian is due entirely to the absence of the $A^2$ terms from the interaction Hamiltonian.

Consider the well-studied Hamiltonian

$$H_1 = \frac{\hbar \omega_{ab}}{2} \sum_{j=1}^{N} \sigma_j^z + \hbar \omega a^+ a + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} (\sigma_j^+ a + a^+ \sigma_j).$$

(1)

This Hamiltonian describes the collective interaction of a single mode of radiation (frequency $\omega$) with a single transition between levels $a$ and $b$ (frequency $\omega_{ab} > 0$) in $N$ identical two-level atoms. Operators $a$ and $a^+$ denote here the annihilation and creation operators of the photons; $\sigma_j^x$, $\sigma_j^y$, $\sigma_j^z$ are Pauli matrices used to describe the $j$th atom. The Hamiltonian (1), sometimes called the Dicke Hamiltonian, may be derived from the more familiar one

$$H = \sum_{j=1}^{N} \left[ \frac{1}{2m} \left( \mathbf{p}_j - e \mathbf{A}(\mathbf{r}_j) \right)^2 + V(\mathbf{r}_j) \right] + \hbar \omega a^+ a$$

(2)

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Dicke Model Quantum Phase Transition ($T=0$)


\[ H_{\text{Dicke}} = \omega a^+ a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}} (a + a^+)(J^- + J^+) \]

- Holstein-Primakoff representation of angular momentum operators
  \[ J^- = \left( \sqrt{N - b^+ b} \right) b, \quad J_z = b^+ b - \frac{N}{2}, \quad [b, b^+] = 1 \]

- Large-$N$ expansion of $H_{\text{Dicke}}$
  \[ \rightarrow H_{\text{normal}}, \quad H_{\text{SR}} \text{ quadratic in } (a, a^+, b, b^+) \]
  \[ \rightarrow \text{diagonalise (Bogoliubov transformation)} \]
  \[ \rightarrow \text{excitation energies} \]
Excitation Energies \( (\omega = \omega_0 = 1, \quad \lambda_c = 0.5) \)
Note: Derivation of \( \{H_{\text{normal}}, H_{\text{SR}}\} \)

\[
\begin{align*}
\lambda < \lambda_c & : \quad J^- \to \sqrt{N} b \\
\lambda > \lambda_c & : \quad a \to a \pm \alpha, \quad b \to b \pm \beta \text{ (coherent displacements)} \\
& \quad \text{then expand in } N \\
& \quad \text{(i.e. linearisation about semiclassical amplitudes)} \\
& \quad \langle a^+ a \rangle = |\alpha|^2, \quad \langle J_z \rangle = |\beta|^2 - \frac{N}{2}
\end{align*}
\]
Note: Ground State “Wave Function”

\[ |\psi(x, y)|^2 \quad (N = 10 \text{ atoms}) \]

Transition from localised state to delocalised “Schrödinger Cat” state

\[ |\Psi_g \rangle \sim |\alpha \rangle -N/2 \rangle_x + |-\alpha \rangle N/2 \rangle_x \]

where

\[ J_x |\pm N/2 \rangle_x = \pm N/2 |\pm N/2 \rangle_x \]
Entanglement properties

Critical behaviour of atom-field and atom-atom quantum entanglement at transition

Issues to confront:

- To date, $\lambda << \{\omega, \omega_0\}$ in cavity QED experiments
- Atomic spontaneous emission, cavity mode losses
- And the $A^2$ issue

Our approach:

- Raman scheme, $\{\omega, \omega_0\} \propto \{$level shifts, Raman detunings$, \}$,
  $\lambda \propto$ Raman transition rate
- Open-system dynamics
  $\Rightarrow$ non-equilibrium (dynamical) quantum phase transition

Dimer, Estienne, Parkins & Carmichael, PRA 75, 013804 (2007)
Possible Realisation in Optical Cavity QED

- $N$ atoms identically coupled to single optical (ring) cavity mode
- Lasers + cavity field drive two distinct Raman transitions between stable ground states $|0\rangle$ and $|1\rangle$
Model: Adiabatic elimination of atomic excited states

Effective Hamiltonian (rotating frame)

\[ H = \left[ \delta_{\text{cav}} + \frac{1}{2} N \left( \frac{g_r^2}{\Delta_r} + \frac{g_s^2}{\Delta_s} \right) \right] a^+ a + \left( \frac{g_r^2}{\Delta_r} - \frac{g_s^2}{\Delta_s} \right) a^+ a J_z 
+ \left( \frac{\Omega_r^2}{4\Delta_r} - \frac{\Omega_s^2}{4\Delta_s} + \delta' \right) J_z 
+ \frac{g_r\Omega_r}{2\Delta_r} (aJ^+ + a^+J^-) + \frac{g_s\Omega_s}{2\Delta_s} (a^+J^+ + aJ^-) \]

Choose \[
\frac{g_s^2}{\Delta_s} = \frac{g_r^2}{\Delta_r}, \quad \frac{g_r\Omega_r}{2\Delta_r} = \frac{g_s\Omega_s}{2\Delta_s} \]

then …
Effective (Dissipative) Dicke Model

Master equation for atom-field density operator $\rho$:

$$\dot{\rho} = -i[H, \rho] + \kappa \big(2\rho a^+ - a^+\rho - \rho a^+a\big)$$

where

$$H = \omega a^+a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}(a + a^+)(J^- + J^+)$$

with

$$\omega = \delta_{\text{cav}} + \frac{Ng_r^2}{\Delta_r}, \quad \omega_0 = \delta', \quad \lambda = \frac{\sqrt{N}g_r\Omega_r}{2\Delta_r}$$

"tunable" such that $\omega \sim \omega_0 \sim \lambda$
Potential Experimental Parameters?

- **Ring cavity / many atoms** (e.g., Tübingen, Hamburg, \(^{85}\text{Rb}\))
  \[
g_i/2\pi \approx 50 \text{ kHz}, \quad \kappa/2\pi \approx 20 \text{ kHz}, \quad N \approx 10^6
  \]
  \[
  \frac{\Omega_i}{\Delta_i} \approx 0.005 \quad \Rightarrow \quad \frac{\lambda}{2\pi} \approx \sqrt{N} \times 0.125 \text{ kHz} \approx 125 \text{ kHz}
  \]

- **Strong coupling CQED / few atoms** (e.g., Georgia Tech, \(^{87}\text{Rb}\))
  \[
g_i/2\pi \approx 30 \text{ MHz}, \quad \kappa/2\pi \approx 2 \text{ MHz}, \quad N \approx 100
  \]
  \[
  \frac{\Omega_i}{\Delta_i} \approx 0.05 \quad \Rightarrow \quad \frac{\lambda}{2\pi} \approx \sqrt{N} \times 0.75 \text{ MHz} \approx 7.5 \text{ MHz}
  \]
e.g., ring cavity + $^{87}\text{Rb}$ + magnetic field
Holstein-Primakoff Analysis ($N \rightarrow \infty$): Normal Phase

\[
\dot{\rho} = -i[H^{(1)}, \rho] + \kappa \left( 2\alpha \rho a^+ - a^+ a \rho - \rho a^+ a \right)
\]

with

\[
H^{(1)} = \omega a^+ a + \omega_0 b^+ b + \lambda (a + a^+)(b + b^+)
\]

for

\[
\lambda < \lambda_c = \frac{1}{2} \sqrt{\frac{\omega_0}{\omega}} \left( \kappa^2 + \omega^2 \right)
\]

\[
\kappa = 0.1, 0.2, 0.5
\]
Holstein-Primakoff Analysis ($N \to \infty$): Superradiant Phase

\[ a \to c \pm \alpha, \quad b \to d \mp \beta \]

\[
\dot{\rho} = -i[H^{(2)}, \rho] + \kappa \left( 2c \rho c^+ - c^+ \rho c - \rho c^+ c \right)
\]

with

\[
H^{(2)} = \omega c^+ c + \frac{\omega_0}{2\mu} (1 + \mu) d^+ d + \frac{\omega_0(1-\mu)(3+\mu)}{8\mu(1+\mu)} (d^+ + d)^2
\]

\[ + \lambda \mu \sqrt{\frac{2}{1+\mu}} (c^+ + c)(d^+ + d), \quad \mu = \frac{\lambda^2_c}{\lambda^2} \]

for

\[
\lambda > \lambda_c = \frac{1}{2} \sqrt{\frac{\omega_0}{\omega}} (\kappa^2 + \omega^2)
\]
Field and Atomic Amplitudes $\alpha$ and $\beta$

\[
\alpha = \pm \frac{\lambda \omega_0}{2 \lambda_c^2} \sqrt{\frac{N}{4} \left(1 - \frac{\lambda^4}{\lambda^2}\right)} \left(1 + i \frac{\kappa}{\omega}\right), \quad \beta = \mp \sqrt{\frac{N}{2} \left(1 - \frac{\lambda_c^2}{\lambda^2}\right)}
\]

Graphs showing field amplitude $\alpha / N^{1/2}$ and atomic coherence $\beta / N^{1/2}$ as functions of $\lambda$, with $\omega = \omega_0 = 1$ and $\kappa = 0.1$. 
Spectra of the Light Emitted from the Cavity

Cavity fluorescence

Probe transmission
Eigenvalues of the linearised model
Probe transmission spectra \((\omega = \omega_0 = 1, \kappa = 0.2)\)
Homodyne detection (quadrature fluctuation spectra)
Atom-field entanglement

Gaussian continuous variable state: quadrature/EPR operators

\[ X^\theta_a = \frac{1}{2}(ae^{-i\theta} + a^+e^{i\theta}), \quad X^\phi_b = \frac{1}{2}(be^{-i\phi} + b^+e^{i\phi}) \]

\[ u = X^\theta_a + X^\phi_b, \quad v = X^\theta_a + \pi/2 - X^\phi_b + \pi/2 \]

Possible to deduce from cavity output field

\[ \langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle < 1 \]
Other possibilities for effective spin systems

- Two cavity modes + off-resonant Raman transitions
- Effective spin-spin interactions:

\[ H_{\text{eff}} = -2hJ_z - \frac{2\lambda}{N} (J_x^2 + \gamma J_y^2), \quad -1 < \gamma < 1 \]

(Lipkin-Meshkov-Glick model)

- \( \lambda \gg \text{dissipative rates possible} \)
- 1\text{st} or 2\text{nd} order quantum phase transitions
Example:

\[ H_{\text{eff}} = -2hJ_z - \frac{2\lambda}{N} J_x^2, \quad \lambda < 0 \]

\[
\dot{\rho} = -i[H_{\text{eff}}, \rho] + \frac{4\Gamma_a}{N} (2J_x \rho J_x^2 - J_x^2 \rho - \rho J_x^2) + \frac{\Gamma_b}{N} (2J_- \rho J_+ - J_+ J_- \rho - \rho J_+ J_-)
\]

1st-order quantum phase transition

\[ \langle J_z \rangle / N/2 \]

\[ \lambda = -1, \quad \Gamma_a = 0.01, \quad \Gamma_b = 0.2 \]

Probe transmission spectrum

\( h = 0.8 \)

\( h = 0.01 \)
Bipartite entanglement criterion / spin variances

\[ C_\varphi = 1 - \frac{4}{N} \langle \Delta J_x^2 \rangle - \frac{4}{N^2} \langle J_\varphi \rangle^2 > 0, \quad J_\varphi = J_x \sin \varphi + J_y \cos \varphi \]

\[ C_R = \max_\varphi C_\varphi \]
Time-dependence of entanglement, \( C_R(t) \)

Note: Cavity output field \( b_{\text{out}} \propto J_- \) so \( C_R \) can be deduced from measurable correlation functions
Summary

- Proposed realisation of Dicke model in cavity QED for study of (non-equilibrium) quantum phase transition
- Well-defined cavity output provides measurable signatures/properties of the phase transition
- Other effective spin models possible
Further possibilities …

• Finite-$N$ systems
  – small $\rightarrow$ large quantum noise
  – entangled state preparation
    and characterisation
  – measurement back-action

• Combination with optical-lattice many-body systems
  (long-range + short-range interactions)
• Disordered systems